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A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. It should be remembered that, at the present stage of mathematical research, hardly any paper is likely to be so completely original as to be independent of earlier work in the same direction; and that readers are often helped to appreciate the importance of a new investigation by seeing its connection with more familiar results.

The principal results of a paper should, when possible, be enunciated separately and explicitly in the form of definite theorems.

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THE PRESIDENCY COLLEGE,
MADRAS.
16th February 1924.

G. A. SRINIVASAN.
Honorary Assistant Secretary.

$2n$ points n on each face. The first set of rotations carry each face into itself while the second interchange the 2 faces. These 2 sets of n points are the roots of two corresponding polynomials of the pencil in involution. Here too as in the previous case we find 2 perfect nth powers, namely, the centres of the 2 faces each taken n times over, but these are *not* the fixed members of the involution but are carried each into the other by I. The fixed members of I must have their roots on the edges of the dihedron (since they lie on both the faces) and are in fact represented by the vertices and the mid. edges of the polygon. Hence,

The necessary and sufficient condition for a dihedral break-up is that the pencil contains two perfect nth powers which are interchanged by the involution. ... ... (21)

The sufficiency of the condition is easily verified.

13. Examining the other polyhedra, we find only for the octahedron a group of rotations carrying an arbitrary point into 2 imprimitive sets of n each. In fact, if we paint the faces of an octahedron so that no two faces which have an edge in common are of the same colour, we find that 2 colours, say black and white, suffice. Every rotation which carries the octahedron into itself must either interchange the black and white faces, or carry each set into itself as adjacent faces remain adjacent in the new position of the solid. Transferring the points on the sphere to the inscribed octahedron by central projection, we shall have an arbitrary point on a face carried on into a configuration of 24 points lying in 3's on the eight faces, the 3 points in each face being arranged symmetrically about the centre of that face. We have, thus, a configuration of 12 white and 12 black points which are respectively the roots of 2 polynomials of the pencil which are exchanged by the involution I. Four faces of the same colour, when produced, form a tetrahedron and the group of 12 rotations determined by it carries each of the 2 tetrahedra into itself. The remaining 12 rotations of the octahedral group carry each point into one of a different colour and hence correspond to the n-homographies which interchange the 2 polynomials. The fixed members of this involution correspond to the coincidence of 12 white and 12 black points and this can occur only at an edge or a vertex.

A point near one of the vertices of the solid is carried over into 6 sets of points clustering round the 6 vertices, each set consisting of 2 white and 2 black points. Hence if $\psi(x)$ is the sextic whose roots are the 6 vertices, $\{\psi(x)\}^2$ is one of the fixed members of the pencil. The other fixed member is evidently the 12-ic representing the 12 mid. edge points, for at each of these one black and one white point come together.

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The only other configuration of some interest is that of the centres of the 8 faces at each of which 3 points of the same colour cluster together. Hence the pencil contains 2 perfect cubes which are interchanged by the involution.

Hence *we see that* $f(x)g(y) + g(x)f(y)$ *can be factorised into bilinear factors only when the pencil* $f(x) + \lambda g(x)$ *either*

(*i*) *contains 2 perfect* n^{th} *powers* $f(x)$ *and* $g(x)$.

(*ii*) *contains 2 perfect* n^{th} *powers* $f(x)+kg(x)$ *and* $f(x)-kg(x)$.

or (*iii*) *consists of polynomials belonging to a tetrahedral group of linear substitutions.* (22)

The Orthoptic Locus.

14. Let C be a rational curve whose reduced class (the number of tangents in a given direction) is n. Then the equation of the tangent at any point "t" on the curve is of the form

$$x f(t)\, \phi(t) + y\, g(t)\, \phi(t) + h(t) = 0 \qquad \ldots \quad (23)$$

where f and g are n-ics in t.

This tangent is parallel to

$$y = \lambda x \qquad \ldots \quad \ldots \quad (24)$$

provided

$$f(t) + \lambda g(t) = 0 \qquad \ldots \quad \ldots \quad (25)$$

The parameters of 2 points the tangents at which are perpendicular are respectively the roots of two equations P_λ and P_μ of the pencil (25) connected by the involution

$$\lambda \mu + 1 = 0 \qquad \ldots \quad \ldots \quad (26)$$

The points given by each equation form a set of associates, a point corresponding to a root of one equation having as its conjugates all the points given by the other. If with each of such a pair of points we associate the intersection of their tangents, we obtain the orthoptic locus which is thus seen to be a derived locus of the second type.

We have seen that such an involutric relation between the members of a polynomial pencil splits up completely into homographic relations in 3 three different cases and we have accordingly 3 different types of degeneracy of the orthoptic locus.

15. For a cyclic break-up, the fixed members of the involution must be perfect nth powers, *i.e.* the n tangents from either of the circular points at infinity to the curve coalesce. The curve has then a unique singular focus if the tangents from I and J are distinct, and no focus at all if both the tangents coincide with the line at infinity.

An interesting property of such curves, which we shall find useful here, is the following :—

If the n tangents from I or J to a curve whose reduced class is n coalesce, a tangent to the curve turns through a total angle $n\pi$ as the point of contact travels once completely round the curve.

Taking α and β to be the parameters of the circular points,* we have, in (23)

$$\left.\begin{aligned} f(t) + ig(t) &= (t-\alpha)^n \\ f(t) - ig(t) &= (t-\beta) \end{aligned}\right\} \quad \ldots \qquad \ldots (27)$$

Hence

$$\sqrt[n]{\frac{f(t) + ig(t)}{f(t) - ig(t)}} = \frac{t-\alpha}{t-\beta} \qquad \ldots \qquad \ldots (28)$$

being a linear transform of t, is a possible parameter for the curve, *i.e.* the coordinates can be expressed as rational functions of this parameter. Now if ψ denotes the inclination of the tangent to the axis of x, the first member of (28) is easily seen to be a numerical multiple of $e^{\frac{2i\psi}{n}}$. Hence $\tan\frac{1}{n}\psi$ is also a possible parameter. As the representative point travels completely round the curve C, the parameter $t = \tan\frac{1}{n}\psi$ passes through its range of values from $-\infty$ to $+\infty$ and hence ψ increases by $n\pi$.

Consider now a tangent of inclination ψ. There are n tangents perpendicular to it, namely, those of inclination

$$\psi + \frac{\pi}{2},\ \psi + \frac{3\pi}{2}, \qquad \ldots \qquad \ldots\ \psi + \frac{(2n-1)\pi}{2}.$$

These n intersections arrange themselves on the following loci.

A locus through the meets of tangents of inclination

$$\left\{\begin{aligned} &\left(\psi,\ \psi + \frac{\pi}{2}\right) \\ \text{and}\ &\left(\psi,\ \psi + \frac{(2n-1)\pi}{2} = \psi - \frac{\pi}{2}\right) \end{aligned}\right\} \quad L_1\,;$$

one through the meets of tangents of inclination $\qquad \ldots \quad (29)$

$$\left\{\begin{aligned} &\left(\psi,\ \psi + \frac{3\pi}{2}\right) \\ \text{and}\ &\left(\psi,\ \psi + \frac{2n-3}{2}\pi = \psi - \frac{3\pi}{2}\right) \end{aligned}\right\} \quad L_2\,;$$

... ...

* If the curve be non-focal, $\varphi(t)$ vanishes for $t = \alpha$ and $t = \beta$; while if C is unifocal, φ does not vanish for these values.

There are $\frac{n}{2}$ or $\frac{n+1}{2}$ partial orthoptics according as n is even or odd, the last locus in the second case passing through the single point

$$\psi, \; \psi + \frac{n\pi}{2}.$$

These are respectively the loci of the meets of tangents at t and t_1 connected by

$$\frac{t_1 - t}{1 + tt_1} = \tan\frac{\pi}{2n}, \; \tan\frac{3\pi}{2n}, \; \tan\frac{5\pi}{2n}, \quad \ldots \quad (30)$$

the simpler relations into which the general $n \cdot n$ relation of conjugacy between t and t_1 resolves itself in this case.

16. Epi- and Hypo-cycloids are the most familiar curves of the type here discussed. Thus the 3 cusped Epi-cycloid which is the envelope of

$$x \sin\psi - y\cos\psi = \sin\frac{3}{5}\psi \quad \ldots \quad \ldots \quad (31)$$

is a curve of class 5 for which the origin is the quintuple focus. The orthoptic locus consists of 2 Epi-trochoids of order 8 which are the loci L_1 and L_2 of (30), and a circle,*

$$x^2 + y^2 = a^2 \quad \ldots \quad \ldots \quad (32)$$

on which lie the intersection of (31) with

$$x\cos\psi + y\sin\psi = -\cos\frac{3}{5}\psi \quad \ldots \quad (33)$$

which is of inclination $\psi + \frac{5\pi}{2}$.

The circle (32) must be counted twice †, since the point

$$x = -\cos\frac{8}{5}\psi, \quad y = -\sin\frac{8}{5}\psi,$$

which is the intersection of (31) and (33) is unaltered by replacing ψ by

---

* In his *Cubic and Quartic Curves*, pp. 213—214, Basset states that the orthoptic locus of an Epi-cycloid is an Epi-trochoid. Basset's Epi trochoid is our locus L_1 and is hence but a part of the locus. The complete orthoptic consists of $\frac{n}{2}$ such Epi-trochoids, or $\frac{n-1}{2}$ Epi-trochoids and a circle repeated $\frac{n-1}{2}$ times, according as n (the class of the Epi-cycloid) is even or odd.

† *Vide*: E. Bally, *Geometrie Synthetique des unicursales de troisieme class et de quatrieme order*, p. 5.

$\psi + \frac{5\pi}{4}$, thus showing that from each point of the circle can be drawn 2 pairs of orthogonal tangents to C. All these partial loci taken together constitute a 10-circular 20-*ic* which is what the orthoptic of a curve of class 5 should be.

Dihedral Break-up.

17. We have seen that, for a dihedral break-up, the pencil $f+\lambda g$ should contain 2 perfect powers which are interchanged by the involution, *i.e.* that all the tangents in 2 perpendicular directions coalesce. Choosing the axes of co-ordinates in these directions, we see that $f(t)$ and $g(t)$ in (27) must be perfect nth powers, so that $\sqrt[n]{\frac{-f(t)}{g(t)}} = (\tan\psi)^{\frac{1}{n}}$ is a possible parameter. The curves $y = x^n$ come under this category.

Taking $\quad f(t) = t^n, \; g(t) = 1 \quad \ldots \quad \ldots \quad \ldots \quad (34)$

in (27) we have for the parameters of tangents in two perpendicular directions

$$\left.\begin{matrix} t & w^2 t & w^4 t & w^6 t & w^{2n-2} t \\ \frac{w}{t} & \frac{w^3}{t} & \frac{w^5}{t} & \frac{w^7}{t} & \frac{w^{2n-1}}{t} \end{matrix}\right\} \quad \ldots \quad \ldots \quad (35)$$

w being a primitive $2n^{\text{th}}$ root of unity. These are seen to be the roots of 2 equations in λ and μ of the form (25) connected by the relation of perpendicularity (26).

We have therefore the following partial loci.

Corresponding to the relation

$$t_1 = \frac{w}{t}, \text{ a locus } L_1 \text{ through the } n \text{ points } \left(w^{2r} t, \frac{w^{2r+1}}{t}\right);$$

$$t_1 = \frac{w^3}{t} \; \ldots \text{ locus } L_2 \text{ through the } n \text{ points } \left(w^{2r} t, \frac{w^{2r+3}}{t}\right) \; \ldots \quad (36)$$

The n^2 intersections given by (35) thus lie in n's on the n loci (36).

Octahedral Break-up.

18. Finally for an octahedral break-up it is necessary that each member λ of the 12-*ic* pencil (25) be invariant for the substitutions of a tetrahedral group and be carried into another μ given by (26) for the remaining substitutions of the octahedral group. These conditions are satisfied, for instance, if

$$f(t) = t^4 + 2\sqrt{3}\, i\, t^2 + 1$$

and

$$g(t) = i\,(t^4 - 2\sqrt{3}\, i\, t^2 + 1).$$

There are 12 substitutions carrying each point into a conjugate, 6 of order 2, and 6 of order 4. Each of the former pairs a tangent in one direction uniquely with one in a perpendicular direction and hence passes through 12 out of the 144 intersections of tangents in these 2 directions. We have therefore 6 loci L_r each through 12 intersections. The other partial loci correspond to the 3 substitutions H of order 4 and their 3 inverses. But as the same locus is given by a certain relation and its inverse we have but 3 loci altogether. With reference to H, the 24 points of contact we are considering, divide themselves into 6 sets of 4

$$\left.\begin{matrix} t_r, \; H^2 t_r \\ Ht_r, \; H^3 t_r \end{matrix}\right\} \quad \begin{matrix} r = 1, 2, 3, 4, 5, 6 \\ H^4 \equiv 1 \end{matrix}$$

the tangents at points in the same row being parallel and at points in different rows perpendicular. We have thus 6 sets of 4 intersections on each of the 3 loci given by

	Intersections		Locus
$t_1 = H^1(t)$	24	...	L_7
$t_1 = H^2(t)$	24	...	L_8
$t_1 = H^3(t)$	24	...	L_9

These 9 loci account altogether for all the 144 intersections and together constitute a 66-circular 132-*ic*, the complete orthoptic of a curve of class 12.

On the Cross Ratio of Four Point-Groups of an Involution

By R. Vaidyanathaswamy.

The set of point-groups which are roots of polynomials of the form, $f(x) - \lambda \phi(x)$, [where $f(x)$, $\phi(x)$ are of order n, and λ is an arbitrary parameter], is often called *a generalised involution of order n*. An examination of the cross ratio of four point-groups of a generalised involution, will reveal the principle upon which depends the expression given by Mr. R. Gopalaswamy (*J. I. M. S.*, Dec. 1922: 'Pencils of Conics') for the cross ratio of four conics of a pencil—or, what is the same thing, of four point-pairs of an Involution of order 2.

I. Consider four groups of a generalised involution, corresponding to the values $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ of the parameter λ.

Let the roots of $[f(x) - \lambda_r \phi(x)]$ $(r = 1, 2, 3, 4)$ be α_k, β_k, γ_k, δ_k respectively, $(k = 1, 2\ldots, n)$. Consider the two functions :

$$\frac{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)}{(\lambda_1 - \lambda_4)(\lambda_3 - \lambda_2)} = [\lambda_1 \lambda_2 \lambda_3 \lambda_4] \qquad \ldots \quad (1)$$

$$\Pi \frac{(\alpha_p - \beta_q)(\gamma_r - \delta_s)}{(\alpha_p - \delta_s)(\gamma_r - \beta_q)} = \Pi\, [\alpha_p\, \beta_q\, \gamma_r\, \delta_s], \qquad \ldots \quad (2)$$

where the product extends to all independent values of p, q, r, s, from 1 to n. It is easy to see that the expressions (1) and (2) become zero or infinite together. For instance, if (2) is zero, then either an α is equal to a β or a γ to a δ. But, since the four groups belong to an involution, if *one* α is equal to a β, then the whole set of α's and β's will become identical, so that $\lambda_1 = \lambda_2$ and consequently (1) is zero. A similar proof would shew that (1) and (2) become infinite together.

Hence it would follow that the expression (2) could only be a power of the expression (1). Since the numerator and denominator of (2) are symmetric functions of the α's of degree n^3 in each α, it follows that (2) is a rational function of degree n^3 in λ_1. Hence,

$$\Pi\, [\alpha_p\, \beta_q\, \gamma_r\, \delta\,] = k\, [\lambda_1\, \lambda_2\, \lambda_3\, \lambda_4]^{n^3} \qquad \ldots \quad (3)$$

where k is numerical. By putting $\lambda_1 = \lambda_3$, it immediately becomes evident that $k = 1$.

From this result it appears that it would not in general be possible to express $[\lambda_1 \lambda_2 \lambda_3 \lambda_4]$ rationally in terms of the cross ratios $[\alpha_p \beta_q \gamma_r \delta_s]$.

II. There is however one case, in which it is possible to give a rational expression for $[\lambda_1\lambda_2\lambda_3\lambda_4]$ in terms of the cross ratios $[\alpha_p\beta_q\gamma_r\delta_s]$. This happens when the involution contains two foci (or coalesced groups); or in other words, the pencil $f(x) - \lambda\phi(x)$ contains two perfect n^{th} powers. In this case, it is well-known that the homographic transformation Ω of period n, the fixed points of which are the foci, carries every group of the involution cyclically into itself. It happens then, that the expression

$$\overset{t}{\Pi}\, [\alpha_p, \Omega^t \beta_q, \gamma_r, \Omega^t \delta_s] \; (t=1,2\ldots n)$$

has the same value for any p, q, r, s, this value being in fact equal to $[\lambda_1\lambda_2\lambda_3\lambda_4]$. To prove this, we suppose (by choosing a suitable co-ordinate-system) that the foci are $(0, \infty)$, so that the pencil $f(x) - \lambda\phi(x)$ takes the shape $[x^n - \lambda]$. Then, supposing that $\alpha, \beta, \gamma, \delta$ represent indifferently any one of the α's, β's, γ's, δ's respectively, we have

$$\lambda_1 = \alpha^n, \lambda_2 = \beta^n, \lambda_3 = \gamma^n, \lambda_4 = \delta^n,$$

$$[\lambda_1 \lambda_2 \lambda_3 \lambda_4] = \frac{(\alpha^n-\beta^n)(\gamma^n-\delta^n)}{(\alpha^n-\delta^n)(\gamma^n-\beta^n)} = \overset{\omega}{\Pi}\frac{(\alpha-\omega\beta)(\gamma-\omega\delta)}{(\alpha-\omega\delta)(\gamma-\omega\beta)}$$

$$= \overset{\omega}{\Pi}\, [\alpha, \omega\beta, \gamma, \omega\delta]. \quad (\omega^n = 1) \qquad \ldots (4)$$

This proves our statement. When $n = 2$, it will be noticed that $\Pi\, [\alpha, \omega\beta, \gamma, \omega\delta]$ is the 'super-cross' product of Mr. Gopalaswamy. The result (4) gives the extension of the theorem of the super-cross product, which may be stated thus:

Every group of a generalised involution of the nth order, with two foci is carried cyclically into itself by a homographic transformation Ω of period n, of which the foci are the fixed points. If $\alpha, \beta, \gamma, \delta$ be points belonging to four distinct groups of the involution, the product $\overset{t}{\Pi}\, [\alpha, \Omega^t \beta, \gamma, \Omega^t \delta]$ does not depend on $\alpha, \beta, \gamma, \delta$; but only on the four groups, being in fact equal to their cross ratio.

The Hindu Sine-Table.

By A. A. Krishnaswami Ayyangar.

In the *Surya Siddhanta* and the *Aryabhatiya*, a remarkable recurrence formula is given for the formation of the table of natural sines at intervals of $3°45'$. Aryabhata says:

प्रथमाच्चापज्यार्धाद्यैरूनं खण्डितं द्वितीयार्धं ।
तत्प्रथमज्यार्धांशैस्तैस्तै रूनानि शेषाणि ॥

which may be translated thus:

"Each of the sine-differences is less than the first sine by (two quantities) the difference between the first sine and the preceding sine-difference (called the *dvitiva* sine-difference) as well as by the quotient obtained by dividing (the sum of) all the preceding sine-differences by the first sine."

Let Δ_1 = Aryabhata's sine of $(3\frac{3}{4}°)$ and $\Delta_2, \Delta_3, \Delta_4, \ldots$ the successive sine-differences. Then, Aryabhata's formula is:

$$\Delta_{r+1} = \Delta_1 - (\Delta_1 - \Delta_r) - \frac{\sum_1^r \Delta r}{\Delta_1},$$

or more briefly

$$\Delta r - \Delta_{r+1} = \frac{\sum_1^r \Delta_r}{\Delta_1}.$$

Remembering that Aryabhata's sine

$$= \left(\frac{21600'}{2\pi}\right) \times \text{[the modern sine]}$$

$$= \quad 3438' \quad \times \text{[the modern sine]},$$

Aryabhata's sine of $(3\frac{3}{4}°)$ is equal to $225'$, being the length of the corresponding arc and

$$\sum_1^r \Delta_r = \text{Aryabhata's sine of } (r\alpha), \text{ where } \alpha \equiv 3\tfrac{3}{4}°.$$

The above formula can be written in the form

$$\sin (r+1)\,\alpha - \sin (r\alpha) = \sin (r\alpha) - \sin (r-1)\,\alpha - \frac{\sin (r\alpha)}{225}$$

where r is any integer less than 24.

Mr. G. R. Kaye, in his Article on 'Aryabhata' (*J. A. S. of Bengal*, 1908, Vol. XIX), considers the above formula as a rough attempt at the enunciation or application of Ptolemy's Theorem. But it is just possible the result might have been derived in the manner indicated below:

Let AOB be a quadrant of a circle, and P, Q, R points on the arc AB such that $A\hat{O}P = (r-1)\alpha$, $A\hat{O}Q = r.\alpha$, $A\hat{O}R = (r+1)\alpha$.

Let PR cut OQ at S and P′, Q′, R′, S′ be the projections of P, Q, R, S on OB.

Then $\sum_1^r \Delta_r = OQ'$, $\Delta_{r+1} = Q'R'$, $\Delta_r = P'Q'$,

and $P'S' = S'R'$, since $PS = SR$.

Now $\Delta_r - \Delta_{r+1} = 2\,S'Q'$

$$= 2.\frac{SQ}{OQ}.OQ', \quad \text{since } \frac{S'Q'}{SQ} = \frac{OQ'}{OQ}$$

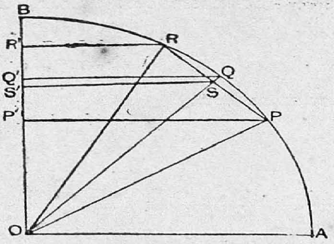
$$= 2.\frac{SQ}{OQ}.\sum_1^r \Delta_r;$$

and $$\frac{2\,SQ}{OQ} = \frac{QR^2}{OQ^2}$$

$$= \left(\frac{225}{3438}\right)^2$$

$$= \frac{1}{225}$$

approximately.



Hence $$\Delta_r - \Delta_{r+1} = \frac{\sum_1^r \Delta_r}{225}$$

$$= \frac{\sum_1^r \Delta_r}{\Delta_1}.$$

Rev. E. Burgess, in his translation of the *Surya Siddhanta*, supposes it to have been obtained empirically. Since the quotients $\dfrac{\sum_1^r \Delta_r}{\Delta_r - \Delta_{r+1}}$ give values varying from 215 to 263 and it is not possible to pitch upon 225 by mere conjecture, we do not agree with him.

Mr. G. R. Kaye in his *Indian Mathematics* (p. 11) observes that the formula is given ostensibly for the formation of the table, but that the table actually given cannot be obtained from the formula. This statement is only partially true, as may be seen from the following procedure for obtaining $\triangle_3$, $\triangle_4$, ...

[*N.B.*—Fractions less than $\frac{1}{450}$ are omitted in our calculations];

$\triangle_1 = 225$; $\triangle_2 = 224$

$\therefore \triangle_3 = 222\frac{1}{225}$; $\triangle_4 - \triangle_3 = -1\frac{224}{225} - \frac{222}{225} = -2\frac{221}{225}$;

$\therefore \triangle_4 = 219\frac{1}{45}$; $\triangle_5 - \triangle_4 = -2\frac{221}{225} - \frac{219}{225} = -3\frac{215}{225}$;

$\therefore \triangle_5 = 215\frac{1}{15}$; $\triangle_6 - \triangle_5 = -3\frac{215}{225} - \frac{215}{225} = -4\frac{205}{225}$;

$\therefore \triangle_6 = 210\frac{7}{45}$; $\triangle_7 - \triangle_6 = -4\frac{205}{225} - \frac{210}{225} = -5\frac{190}{225}$;

$\therefore \triangle_7 = 204\frac{14}{45}$.

Comparing these with Aryabhata's Table

$\triangle_1$	225
$\triangle_2$	224
$\triangle_3$	222
$\triangle_4$	219
$\triangle_5$	215
$\triangle_6$	210
$\triangle_7$	205

we find the agreement close enough except in the case of $\triangle_7$. But the fractional parts neglected in $\triangle_3$, $\triangle_4$, $\triangle_5$, and $\triangle_6$ have accumulated sufficiently to affect $\triangle_7$ and we are therefore justified in including them in $\triangle_7$ which thus becomes $204\frac{126}{225}$, *i.e.*, 205 to the nearest integer, which is Aryabhata's value.

Continuing further, $\triangle_8 - \triangle_7 = -6\frac{170}{225}$.

$\therefore \triangle_8 = 198\frac{55}{225}$ or 198 to the nearest integer.

But Aryabhata-sine $30^\circ = \frac{1}{2} \times 3438 = 1719$ by direct calculation, and $\sum_1^8 \Delta_r = 1718$. So Δ_8 has been corrected by Aryabhata to 199.

Proceeding further, $\Delta_9 - \Delta_8 = -7\frac{144}{225}$;

$\therefore \quad \Delta_9 = 191\frac{81}{225}$; $\quad \Delta_{10} - \Delta_9 = -8\frac{110}{225}$;

$\therefore \quad \Delta_{10} = 182\frac{196}{225}$; $\quad \Delta_{11} - \Delta_{10} = -9\frac{68}{225}$;

$\therefore \quad \Delta_{11} = 173\frac{128}{225}$; $\quad \Delta_{12} - \Delta_{11} = -10\frac{17}{225}$;

$\therefore \quad \Delta_{12} = 163\frac{111}{225}$.

Comparing these again with Aryabhata's Table

Δ_8	199
Δ_9	191
Δ_{10}	183
Δ_{11}	174
Δ_{12}	164

we find the divergence at Δ_{12} which can be explained as before; for

$$\sum_1^{12} \Delta_r = 2430 \text{ according to the formula,}$$

while by direct calculation $\sum_1^{12} \Delta_r =$ Aryabhata-sine 45°

$$= \frac{1}{\sqrt{2}} \times 3438 = 2431$$

(to the nearest integer).

Hence Δ_{12} must have been corrected to 164.

Again, $\Delta_{13} - \Delta_{12} = -10\frac{181}{225}$.

$\therefore \quad \Delta_{13} = 153\frac{44}{225}$; $\quad \Delta_{14} - \Delta_{13} = -11\frac{109}{225}$

$\therefore \quad \Delta_{14} = 141\frac{160}{225}$; $\quad \Delta_{15} - \Delta_{14} = -12\frac{26}{225}$

$\therefore \quad \Delta_{15} = 129\frac{134}{225}$; $\quad \Delta_{16} - \Delta_{15} = -12\frac{156}{225}$

$\therefore \quad \Delta_{16} = 116\frac{203}{225}$.

Pausing here for a moment and comparing these values with Aryabhata's Table

$\triangle_{13}$	154
$\triangle_{14}$	143
$\triangle_{15}$	131
$\triangle_{16}$	119*

we find a divergence as before which may, however, be explained thus:

$$\sum_{1}^{16} \triangle_r = \text{Aryabhata-sine } 60^\circ = \frac{\sqrt{3}}{2} \times 3438 = 2977,$$

while, by the formula,

$$\sum_{1}^{16} \triangle_r = \sum_{13}^{16} \triangle_r + 2431 = 2972\frac{91}{225}$$

differing from the true value by $4\frac{134}{225}$. If this be equally distributed among the four $\triangle$'s, we get, to the nearest integer,

$\triangle_{13}$	154
$\triangle_{14}$	143
$\triangle_{15}$	131
$\triangle_{16}$	118 (119).

Next, continuing our process up to 75°, we get

		Aryabhata's values.
$\triangle_{17}$	$104\frac{176}{125}$	107† (106)
$\triangle_{18}$	$91\frac{22}{225}$	93
$\triangle_{19}$	$77\frac{2}{225}$	79
$\triangle_{20}$	$62\frac{130}{225}$	65

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* $\triangle_{16} = 118$ and not 119 as wrongly given by Aryabhata. Bhaskara in his *Gola-Adhyaya* gives the correct figure 118. Probably it is an error in the manuscript perpetuated in the printed edition of Dr. H. Kern. The printed text reads:

मखि भखि फखि धखि णखि ञखि ङखि हस्झ स्ककि किष्ग श्घकि किघ्व ।

† घ्लकि किग्र हक्य घाहास्त स्ग श्झङ्वल्कप्तफछ कलार्धज्याः ॥

According to Aryabhata, $\triangle_{17} = 106$ instead of 107 which is the correct figure given by Bhaskara. If घाहास्त should be corrected into दाहास्थ, we get the correct values 118 and 107 of Bhaskara. The violation in metre in the latter half of the stanza noted by Dr. H. Kern is also significant in this connection, and may be rectified by the reading.........दहसथ.........

But $\sum_{1}^{20} \Delta_r = 3312 \frac{105}{225}$, which differs from the true value of Arya-bhata-sine of 75°, obtained by direct calculation, by $8\frac{120}{225}$; this being equally distributed among the four Δ's, we get Aryabhata's results.

Completing our table now up to 90°, we find

		Aryabhata's values.
Δ_{21}	$50\frac{63}{225}$	51
Δ_{22}	$35\frac{76}{225}$	37
Δ_{23}	$20\frac{54}{225}$	22
Δ_{24}	$5\frac{12}{225}$	7.

Adding to each of the values in the second column $\frac{1}{4}$ of $6\frac{20}{225}$, *i.e.*, one-fourth of the quantity by which $\sum_{1}^{24} \Delta_r$ differs from its true value 3438 (since Aryabhata-sine of 90° = 3438), we get Aryabhata's results, except in the case of Δ_{21}. We get $\Delta_{21} = 51\frac{171}{225}$ which has to be corrected to 51 instead of 52, since otherwise $\sum_{21}^{24} \Delta_r = 118$ instead of 117.

Thus we see how the rule contained in the recurrence formula might have been worked out with corrections applied to the results on comparison with the well-known values of the sines of the common angles 30°, 45°, 60°, 75°, and 90°.

That the early Hindus must have gone through some such process as the above in the formation of the table is supported by another fact, *viz.*, that the values deviate from the corresponding true values in 9 out of 24 instances—a clear proof to show that each value was not calculated *separately*.

Under these circumstances, we have good grounds to reject the theory that the Hindu sine-table is not of Indian origin and that it is based on Ptolemy's Table of Chords.

Modular Equation of the Third Order

BY M. BHIMASENA RAO.

It is well-known that the modular equation of the third order is*

$$\sqrt{k\lambda} + \sqrt{k'\lambda'} = 1.$$

Writing $\sin x$, $\cos x$, for λ, λ', and $\sin y$, $\cos y$ for k, k', the equation becomes

$$\sqrt{\sin x \sin y} + \sqrt{\cos x \cos y} = 1,$$

and supposing $x > y$, this is easily transformed into†

$$\sin(x + y) = 2 \sin \frac{x - y}{2}.$$

For a given value of k, the modular equation has only two real roots, say λ and μ, greater and less than k respectively. Putting $\sin z$ for μ, we have,

$$\sin(y + z) = 2 \sin \frac{y - z}{2}.$$

The object of this paper is to determine the values of x and z for a given value of $\sin 2y$, by the methods of Elementary Geometry. The rationalised trigonometric form stated above will be found to be best adapted for this purpose. The method of solution by the use of Elliptic Functions is briefly indicated at the end of the paper.

§ 1. Draw a circle having its centre S, on the circumference of a given circle, centre O, and touching a diameter AB of the given circle at P. Let X, Y, be the points of intersection of the circles. Join XY, SA, XA, YA. It is easily seen that AS bisects the angle XAY. If the angles OAX, OAY be denoted by x and y respectively, the angles OAS and SAY are

$$\frac{x + y}{2} \text{ and } \frac{x - y}{2}.$$

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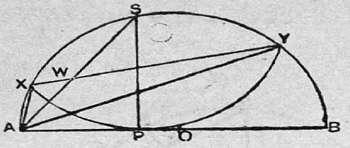
* *Vide* Greenhill : *Elliptic Functions*, page 323 ; Cayley : *Elliptic Functions*, page 208.

† This form was given by the late S. Ramanujam, F.R.S., *vide*, J.I.M.S., Vol. IV., page 78.

If R denotes the radius of the given circle, we have

$$SY = 2R \sin\frac{x-y}{2}, \quad AS = 2R\cos\frac{x+y}{2},$$

$$SP = AS \sin\frac{x+y}{2} = R\sin(x+y).$$



Since SY = SP, we thus have

$$\sin(x+y) = 2\sin\frac{x-y}{2}.$$

We have now obtained the geometrical representation of the modular equation. The inverse of the line XY with respect to the circle (S) is the given circle (O). If SA cuts XY at W, A and W are inverse points. Since P is the point of contact of the tangent from A, we have

$$AP^2 = AW . AS.$$

Because SA bisects the angle XAY,

$$AX . AY = AW . AS,$$

whence $$AP^2 = AX . AY, \quad \ldots \quad \ldots \quad (1)$$

and similarly $$BP^2 = BX . BY. \quad \ldots \quad (1.1)$$

Now $$AP^2 = AS^2 \cos^2\frac{x+y}{2} = 4R^2\cos^4\frac{x+y}{2},$$

and $$AX . AY = 4R^2 \cos x . \cos y.$$

$$\therefore \cos^4\frac{x+y}{2} = \cos x \cos y. \quad \ldots \quad \ldots \quad (1.2)$$

Similarly from the relation $BP^2 = BX . BY$,

we get $$\sin^4\frac{x+y}{2} = \sin x \sin y. \quad \ldots \quad \ldots \quad (1.3)$$

Hence $$\sqrt{\sin x \sin y} + \sqrt{\cos x \cos y} = 1,$$

which is the irrational form of the modular equation.

Again, in the right-angled triangle ASB, SP is perpendicular to AB. We have, therefore,

$$SP^4 = AP^2 . BP^2 = AX . BX . AY . BY ;$$

that is, $$\sin^4(x+y) = 4\sin 2x . \sin 2y, \quad \ldots \quad (1.4)$$

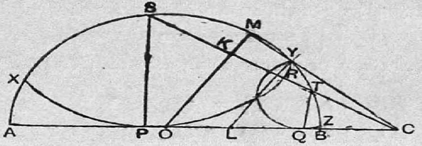
from (1.2) and (1.3).

§ 2. Consider the parabola whose focus is Y, and directrix AB. It passes through S, and cuts the given circle (O) at another point, say T. Let the circle having its centre at T and passing through the focus Y, cut the given circle at Z and touch the directrix AB at Q.

Denoting the angle OAZ by z, we have, as in § 1,

$$\sin(y+z) = 2\sin\frac{y-z}{2}, \qquad \ldots \quad (2)$$

$$\sin^4\frac{y+z}{2} = \sin y \, . \sin z, \qquad \ldots (2.1)$$



$$\cos^4\frac{y+z}{2} = \cos y \, . \cos z, \qquad \ldots (2.2)$$

and $$\sin^4(y+z) = 4\sin 2y \, . \sin 2z. \qquad \ldots (2.3)$$

We now establish a fundamental property of the figure, *viz.*

If M is the middle point of the arc ST, the system (AB, ST, MY) is in involution. ... ... (2.4)

Proof: Since arc SM = arc MT, angle MYS = SAM = MAT = supplement of MYT.

Hence the line MY, being equally inclined to the radii SY, TY of the circles (S), (T), cuts them at equal angles, and passes through the centre of similitude C of the circles. In other words, AB the common tangent, ST the line of centres, and MY are concurrent, and the system (AB, ST, MY) is in involution.

Next, join OM bisecting ST at right angles at K, and draw the radical axis of the circles (S), (T), bisecting the common tangent PQ at L. L is the projection of K. OM being parallel to the radical axis LY, and OKC being a right angle, we see that

$$\text{angle } YLC = MOC = LKC = \theta, \text{ say.}$$

If LY cuts ST at R, we find

$$LR = LK\sin\theta = OK\sin^2\theta,$$

and $$\frac{LY}{LR} = \frac{OM}{OK}.$$

Therefore $LY = R\sin^2\theta$ ... ... (2.5)

Again, in the triangle YOL,

$$\text{angle } YOL = 2y,$$

and

$$\text{angle } YLO = 180^\circ - \theta.$$

Hence

$$\frac{LY}{OY} = \frac{\sin 2y}{\sin\theta} = \sin^2\theta, \text{ from (2.5)}$$

giving,

$$\sin 2y = \sin^3\theta,$$

an important relation ... ... (2.6)

§ 3. We may also obtain in another way the relation

$$\sin 2y = \sin^3\theta.$$

From the involution property of the system (AB, ST, MY) we have,

$$(ABTY) = (BASM),$$

that is,

$$\frac{AB.TY}{BA.SM} = \frac{BT.AY}{AS.BM} = \frac{TA.BY}{SB.AM}.$$

Writing for a moment, α for the angle $OAS = \frac{x+y}{2}$, and β for the angle $OAT = \frac{y+z}{2}$,

whence $\theta = BOM = 2\,OAM = OAS + OAT = \alpha + \beta$,

we easily find that the above relations are replaced by

$$\frac{\sin\beta\,.\cos\beta}{\sin\frac{\alpha-\beta}{2}} = \frac{\sin\beta\,.\cos y}{\cos\alpha\,.\sin\frac{\theta}{2}} = \frac{\cos\beta\,.\sin y}{\sin\alpha\cos\frac{\theta}{2}},$$

giving

$$\sin y = \frac{\sin\alpha\,.\sin\beta\,.\cos\frac{\theta}{2}}{\sin\frac{\alpha-\beta}{2}}, \quad \ldots \quad \ldots \quad (3)$$

and

$$\cos y = \frac{\cos\alpha\,.\cos\beta\,.\sin\frac{\theta}{2}}{\sin\frac{\alpha+\beta}{2}}. \quad \ldots \quad \ldots (3.1)$$

Now the circles (S) and (T) cut at the angle $SAT = \alpha - \beta$. The common tangent PQ is $ST.\sin\theta$. Also from a well-known result,

$$PQ^2 = 4\,SP.TQ.\cos^2\frac{\alpha-\beta}{2}$$

$$= 16\,R^2\sin\alpha.\cos\alpha.\sin\beta.\cos\beta.\cos^2\frac{\alpha-\beta}{2},$$

and

$$ST^2 = 4R^2\sin^2(\alpha-\beta).$$

Hence $$\sin^2\theta = \frac{\sin\alpha \,.\, \cos\alpha \,.\, \sin\beta \,.\, \cos\beta}{\sin^3 \frac{\alpha - \beta}{2}}. \qquad \ldots \qquad \ldots (3.2)$$

From (3), (3.1) and (3.2), we get as before,

$$\sin 2y = \sin^3\theta,$$

§ 4. Next, multiply the numerator and denominator of the right-hand side expression in (3) by $2\sin\frac{\theta}{2} = 2\sin\frac{\alpha+\beta}{2}$, and we get

$$\sin y = \frac{\sin\alpha \sin\beta \sin\theta}{\cos\beta - \cos\alpha};$$

that is, $$\cot\beta \operatorname{cosec}\alpha - \cot\alpha \operatorname{cosec}\beta = \operatorname{cosec} y \sin\theta; \qquad \ldots \qquad (4)$$

and similarly from (3 . 1), we have,

$$\tan\alpha \sec\beta - \tan\beta \,.\, \sec\alpha = \sec y \,.\, \sin\theta. \qquad \ldots (4.1)$$

Also $\sin(\alpha+\beta) = \sin\theta = (\sin 2y)^{\frac{1}{3}}$

may be written $$\sin\alpha\cos\beta + \cos\alpha\sin\beta = (\sin 2y)^{\frac{1}{3}} \qquad \ldots (4.2)$$

These, in conjunction with the results proved in §§ 1 and 2, *viz.*,

$$\sin^4\alpha = \sin x \,.\, \sin y, \ \cos^4\alpha = \cos x \cos y,$$

and $$\sin^4\beta = \sin y \,.\, \sin z, \ \cos^4\beta = \cos y \cos z;$$

give respectively the interesting relations

$$(\tfrac{1}{2}\cot z \operatorname{cosec} x)^{\frac{1}{4}} - (\tfrac{1}{2}\cot x \operatorname{cosec} z)^{\frac{1}{4}} = (2 \operatorname{cosec} y \cot y)^{\frac{1}{12}}, \ \ldots (4.3)$$

$$(\tfrac{1}{2}\tan x \ \sec z)^{\frac{1}{4}} - (\tfrac{1}{2}\tan z \ \sec x)^{\frac{1}{4}} = (2 \ \tan y \sec y)^{\frac{1}{12}}, \ \ldots (4.4)$$

and $$(\tfrac{1}{2}\sin x \ \cos z)^{\frac{1}{4}} + (\tfrac{1}{2}\cos x \ \sin z)^{\frac{1}{4}} = (2 \ \sin y \cos y)^{\frac{1}{12}}, \ \ldots (4.5)$$

The last of these is due to the late Mr. S. Ramanujam, F.R.S.*

§ 5. In the triangle MOY,

the angle MOY = MOB — YOB = $\theta - 2y$.

The perpendicular from Y on OM $= R\sin(\theta - 2y) = KR = LK\cos\theta = OK\sin\theta \,.\, \cos\theta$.

But $OK = R\cos SAT = R\cos\left\{\frac{x+y}{2} - \frac{y+z}{2}\right\}$.

Hence, we have

$$\cos\left\{\frac{x+y}{2} - \frac{y+z}{2}\right\} = \frac{\sin(\theta - 2y)}{\sin\theta \,.\, \cos\theta}; \qquad \ldots (5)$$

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* *Vide* page 78, Vol. IV, J.I.M.S.

we also have

$$\sin\left\{\frac{x+y}{2}+\frac{y+z}{2}\right\}=\sin\Theta=(\sin 2y)^{\frac{1}{3}}.$$

Putting $\sin\Theta=\Omega$, we get

$$\frac{x+y}{2}+\frac{y+z}{2}=\sin^{-1}\Omega,$$

and
$$\frac{x+y}{2}-\frac{y+z}{2}=\cos^{-1}(\sqrt{1+\Omega^2+\Omega^4}-\Omega^2),$$

whence $x=\sin^{-1}\Omega+\cos^{-1}(\sqrt{1+\Omega^2+\Omega^4}-\Omega^2)-\frac{1}{2}\sin^{-1}\Omega^3$, (5.1)

and $z=\sin^{-1}\Omega-\cos^{-1}(\sqrt{1+\Omega^2+\Omega^4}-\Omega^2)-\frac{1}{2}\sin^{-1}\Omega^3$, (5.2)

Another form conv ient for calculation is

$$x+y=\sin^{-1}\Omega-\sin^{-1}\omega\Omega-\sin^{-1}\omega^2\Omega, \quad \ldots (5.3)$$

and
$$y+z=\sin^{-1}\Omega+\sin^{-1}\omega\Omega+\sin^{-1}\omega^2\Omega, \quad \ldots (5.4)$$

where ω is a cube root of unity.

The last one gives, when expanded,

$$y+z=\frac{1}{2}\sin 2y+\frac{1.3.5.7}{2.4.6.8}\frac{\sin^3 2y}{3}+\ldots\ldots$$
$$+\frac{1.3.5\ldots(6n-5)}{2.4.6\ldots(6n-4)}\frac{\sin^{2n-1}2y}{2n-1}+ \quad \ldots (5.5)$$

and when y is small,

$$z=\frac{\sin^3 2y}{128}+\frac{9\sin^5 2y}{2048}, \text{ approximately} \quad \ldots (5.6)$$

As a simple illustration, we may take $\frac{1}{8}$ as the value of $\sin 2y$. Now $\Omega=\frac{1}{2}$; and the preceding results give

$$x=\frac{\pi}{6}+\cos^{-1}\frac{\sqrt{21}-1}{4}-\frac{1}{2}\sin^{-1}\frac{1}{8},$$
$$z=\frac{\pi}{6}-\cos^{-1}\frac{\sqrt{21}-1}{4}-\frac{1}{2}\sin^{-1}\frac{1}{8},$$

and $2y=\sin^{-1}\frac{1}{8}$.

Now, $\sin^{-1}\frac{1}{8}=7^\circ\ 10'\ 50''\cdot 7$

$$\cos^{-1}\frac{\sqrt{21}-1}{4}=\cos^{-1}.8956439$$
$$=26^\circ\ 24'\ 31''.5$$

giving
$$x=52^\circ\ 49'\ 6''.1$$
$$2y=7^\circ\ 10'\ 50''.7$$
$$z=0^\circ\ 0'\ 3''.2$$

The approximation given in (5.6) may, in this case, be written as

$$z = \frac{1}{2^{16}} + \frac{9}{2^{26}} \quad \ldots \quad \ldots \quad \ldots (5.7)$$

$$= \cdot 00001539$$

$$= \text{circular measure of } 3''.2$$

By using the result

$$\sum_{1}^{\infty} (-1)^{n+1} \tan^{-1} \left(e^{-(2n-1)\frac{\pi}{2} \cdot \frac{K'}{K}} \right) = \frac{1}{4} \sin^{-1} k^{*}$$

we obtain,

$$\tan^{-1} e^{-\frac{\pi}{2}\sqrt{63}} - \tan^{-1} e^{-\frac{3\pi}{2}\sqrt{63}} + \ldots\ldots = \frac{z}{4}$$

$$= \frac{\pi}{24} - \tfrac{1}{8} \sin^{-1} \tfrac{1}{8} - \tfrac{1}{4} \cos^{-1} \frac{\sqrt{21} - 1}{4}$$

$$= \frac{1}{2^{18}} + \frac{1}{2^{25}} + \frac{1}{2^{28}}, \text{ approximately;}$$

whence $\pi = -\dfrac{2\sqrt{7}}{21} \log (2^{-18} + 2^{-25} + 2^{-28})$, ... (5.8)

correct to 4 places of decimals.

§ 6. We next express $\sin 2x$ and $\sin 2z$ in terms of $\sin 2y$.

Now $\quad SP = R \sin (x + y)$, and $TQ = R \sin (y + z)$;

also

$$SP \, . \, TQ = SY \, . \, TY$$

$$= 2R \, . \, YR$$

$$= 2R \, . \, (LY - LR)$$

$$= 2R \, (R \sin^2 \theta - LK \sin \theta) \text{ from } (2.5);$$

$$LK = \tfrac{1}{2}(SP + TQ) = R \frac{\sin (\theta - 2y)}{\cos \theta} \text{ from } (5).$$

Hence $SP \, . \, TQ + R \sin \theta \, (SP + TQ) = 2R^2 \sin^2 \theta$. ... (6)

Putting $\sin (x + y) = p \sin 2y = p \, \Omega^3$,

and $\quad \sin (y + z) = q \sin 2y = q \, \Omega^3$,

so that $\quad SP = R p \, \Omega^3$, and $TQ = R q \, \Omega^3$.

we get, $\quad pq \Omega^2 + (p + q) = 2 \Omega^{-2}$, ... (6.1)

and

$$p + q = 2 \{ \tan \theta \, . \cot 2y - 1 \}$$

$$= 2 \{ \sqrt{1 + \Omega^{-2} + \Omega^{-4}} - 1 \}, \quad \ldots (6.2)$$

giving $\quad pq = 2\Omega^{-2} \{ 1 + \Omega^{-2} - \sqrt{1 + \Omega^{-2} + \Omega^{-4}} \}$ (6.3)

* Jacobi : *Fundamenta Nova.*

From (1.4) and (2.3), we have,

$$\sin^4(x+y) = 4\sin 2x \,.\, \sin 2y,$$
$$\sin^4(y+z) = 4\sin 2y \,.\, \sin 2z\,;$$

whence
$$\sin 2x = \frac{p^4.\sin^3 2y}{4}, \quad \ldots \qquad \ldots (6.4)$$

and
$$\sin 2z = \frac{q^4 \sin^3 2y}{4}, \quad \ldots \qquad \ldots (6.5)$$

where p and q are determined from (6.2) and (6.3)

Taking the case of $\sin 2y = \frac{1}{8}$, $\Omega = \frac{1}{2}$; we find,*

$$p+q = 2(\sqrt{21}-1),$$
$$pq = 8(5-\sqrt{21}),$$
$$p = \sqrt{21}-1+\sqrt{(6\sqrt{21}-18)}$$
$$q = \sqrt{21}-1-\sqrt{(6\sqrt{21}-18)}$$
$$\sin 2x = \frac{\{\sqrt{21}-1+\sqrt{(6\sqrt{21}-18)}\}^4}{2^{11}}, \qquad \ldots (6.6)$$
$$\sin 2z = \frac{\{\sqrt{21}-1-\sqrt{(6\sqrt{21}-18)}\}^4}{2^{11}}. \qquad \ldots (6.7)$$

Again † when $\sin 2y = \sqrt{5}-2$, $\Omega = \dfrac{\sqrt{5}-1}{2}$,

$$p+q = 2\sqrt{5},\ pq = 2,$$
$$p = \sqrt{5}+\sqrt{3},\ q = \sqrt{5}-\sqrt{3},$$
$$\sin 2x = \frac{(\sqrt{5}+\sqrt{3})^4(\sqrt{5}-2)^3}{4}, \quad \ldots \qquad \ldots (6.8)$$
$$\sin 2z = \frac{(\sqrt{5}-\sqrt{3})^4(\sqrt{5}-2)^3}{4}, \quad \ldots \qquad \ldots (6.9)$$
$$x = \sin^{-1}\frac{\sqrt{5}-1}{2} + \cos^{-1}\frac{3\sqrt{5}-5}{2} - \tfrac{1}{2}\sin^{-1}(\sqrt{5}-2)$$
$$= \frac{\pi}{4} - \sin^{-1}(\sqrt{5}-2) + \cos^{-1}\frac{3\sqrt{5}-5}{2}$$
$$z = \frac{\pi}{4} - \sin^{-1}(\sqrt{5}-2) - \cos^{-1}\frac{3\sqrt{5}-5}{2}$$

Now, $\sin^{-1}(\sqrt{5}-2) = 13^\circ\ 39'\ 16''$,

$$\cos^{-1}\frac{3\sqrt{5}-5}{2} = 31^\circ\ 20'\ 22'',$$

whence
$$x = 62^\circ\ 41'\ 6'',$$
$$z = \qquad 22''.$$
$$2y = 13^\circ\ 39'\ 16''$$
$$\sin^{-1}\frac{\sqrt{5}-1}{2} = \theta = 38^\circ\ 10'\ 22''.$$

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* *Vide* Greenhill : *Elliptic Functions*, page 331.
† *Vide* Greenhill ; do. do.

These results may be used to evaluate q-products and series.*
For instance,

$$(1+e^{-\pi\sqrt{63}})(1+e^{-3\pi\sqrt{63}})(1+e^{-5\pi\sqrt{63}})\ \ldots\ldots$$
$$= e^{-\frac{\pi\sqrt{7}}{8}}.\,2^{\frac{7}{6}}\{\sqrt{21}-1-\sqrt{(6\sqrt{21}-18)}\}^{-\frac{1}{3}}.$$

$$(1+e^{-\frac{\pi\sqrt{7}}{3}})(1+e^{-\frac{3\pi\sqrt{7}}{3}})(1+e^{-\frac{5\pi\sqrt{7}}{3}})\ \ldots\ldots$$
$$= e^{-\frac{\pi\sqrt{7}}{72}}.\,2^{\frac{7}{6}}\{\sqrt{21}-1+\sqrt{(6\sqrt{21}-18)}\}^{-\frac{1}{3}}$$

$$(1+e^{-\pi\sqrt{45}})(1+e^{-3\pi\sqrt{45}})(1+e^{-5\pi\sqrt{45}})\ldots\ldots$$
$$= e^{-\frac{\pi\sqrt{5}}{8}}.\,2^{\frac{1}{12}}(\sqrt{5}+\sqrt{3})^{\frac{1}{3}}(\sqrt{5}+2)^{\frac{1}{4}}$$

$$(1+e^{-\frac{\pi\sqrt{5}}{3}})(1+e^{-\frac{3\pi\sqrt{5}}{3}})(1+e^{-\frac{5\pi\sqrt{5}}{3}})\ \ldots\ldots$$
$$= e^{-\frac{\pi\sqrt{5}}{72}}.\,2^{\frac{1}{12}}(\sqrt{5}-\sqrt{3})^{\frac{1}{3}}(\sqrt{5}+2)^{\frac{1}{4}}$$

$$\operatorname{sech}\frac{\pi}{2}\sqrt{63}-\tfrac{1}{3}\operatorname{sech}\frac{3\pi}{2}\sqrt{63}+\tfrac{1}{5}\operatorname{sech}\frac{5\pi}{2}\sqrt{63}-\ \ldots\ldots$$
$$=\frac{\pi}{12}-\tfrac{1}{4}\sin^{-1}\tfrac{1}{8}-\tfrac{1}{2}\cos^{-1}\frac{\sqrt{21}-1}{4}$$

$$\operatorname{sech}\frac{\pi}{2}\frac{\sqrt{7}}{3}-\tfrac{1}{3}\operatorname{sech}\frac{3\pi}{2}\frac{\sqrt{7}}{3}+\tfrac{1}{5}\operatorname{sech}\frac{5\pi}{2}\frac{\sqrt{7}}{3}-\ \ldots\ldots$$
$$=\frac{\pi}{12}-\tfrac{1}{4}\sin^{-1}\tfrac{1}{8}+\tfrac{1}{2}\cos^{-1}\frac{\sqrt{21}-1}{4}$$

$$\operatorname{sech}\frac{\pi}{2}\sqrt{45}-\tfrac{1}{3}\operatorname{sech}\frac{3\pi}{2}\sqrt{45}+\tfrac{1}{5}\operatorname{sech}\frac{5\pi}{2}\sqrt{45}-\ldots\ldots$$
$$=\frac{\pi}{8}-\tfrac{1}{2}\sin^{-1}(\sqrt{5}-2)-\cos^{-1}\frac{3\sqrt{5}-5}{2}$$

$$\operatorname{sech}\frac{\pi}{2}\frac{\sqrt{5}}{3}-\tfrac{1}{3}\operatorname{sech}\frac{3\pi}{2}\frac{\sqrt{5}}{3}+\tfrac{1}{5}\operatorname{sech}\frac{5\pi}{2}\frac{\sqrt{5}}{3}-\ \ldots\ldots$$
$$=\frac{\pi}{8}-\tfrac{1}{2}\sin^{-1}(\sqrt{5}-2)+\cos^{-1}\frac{3\sqrt{5}-5}{2}.$$

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* Cayley : *Elliptic Functions*, page 287 ; Greenhill : *Elliptic Functions*, page 286, § 42.

§ 7. We might give another form to the values of x, y, z with the help of the values of $\sin 2x$, $\sin 2y$, $\sin 2z$ given in (6.4) and (6.5). The modular relations

$$\sin(x+y) = 2\sin\frac{x-y}{2},$$

and

$$\sin(y+z) = 2\sin\frac{y-z}{2},$$

are satisfied by taking

$$x = \tfrac{1}{2}\sin^{-1} a\Omega + \sin^{-1}\frac{a\Omega}{2},$$

$$y = \tfrac{1}{2}\sin^{-1} a\Omega - \sin^{-1}\frac{a\Omega}{2},$$

$$= \tfrac{1}{2}\sin^{-1} b\Omega + \sin^{-1}\frac{b\Omega}{2},$$

$$z = \tfrac{1}{2}\sin^{-1} b\Omega - \sin^{-1}\frac{b\Omega}{2},$$

where from (6.4) and (2.3), $a\Omega = (4\sin 2x\,.\,\sin 2y)^{\frac{1}{4}}$,

and $b\Omega = (4\sin 2y\,.\,\sin 2x)^{\frac{1}{4}}$,

while from (6.1) a, b are connected by the simple relation,

$$ab + a + b = 2, \qquad \ldots \qquad \ldots (7)$$

Thus with $\sin 2y = \surd 5 - 2$,

we have,

$$x = \tfrac{1}{2}\sin^{-1}(\surd 5+\surd 3)(\surd 5-2) + \sin^{-1}\frac{(\surd 5+\surd 3)(\surd 5-2)}{2}$$

$$y = \tfrac{1}{2}\sin^{-1}(\surd 5+\surd 3)(\surd 5-2) - \sin^{-1}\frac{(\surd 5+\surd 3)(\surd 5-2)}{2}$$

$$= \tfrac{1}{2}\sin^{-1}(\surd 5-\surd 3)(\surd 5-2) + \sin^{-1}\frac{(\surd 5-\surd 3)(\surd 5-2)}{2}$$

$$z = \tfrac{1}{2}\sin^{-1}(\surd 5-\surd 3)(\surd 5-2) - \sin^{-1}\frac{(\surd 5-\surd 3)(\surd 5-2)}{2}.$$

Similarly with $\sin 2y = \frac{1}{8}$, we have

$$x = \tfrac{1}{2}\sin^{-1}\frac{\surd 21 - 1 + \surd(6\surd 21 - 18)}{8} + \sin^{-1}\frac{\surd 21 - 1 + \surd(6\surd 21 - 18)}{16}$$

$$y = \tfrac{1}{2}\sin^{-1}\frac{\surd 21 - 1 + \surd(6\surd 21 - 18)}{8} - \sin^{-1}\frac{\surd 21 - 1 + \surd(6\surd 21 - 18)}{16}$$

$$= \tfrac{1}{2}\sin^{-1}\frac{\surd 21 - 1 - \surd(6\surd 21 - 18)}{8} + \sin^{-1}\frac{\surd 21 - 1 - \surd(6\surd 21 - 18)}{16}$$

$$z = \tfrac{1}{2}\sin^{-1}\frac{\surd 21 - 1 - \surd(6\surd 21 - 18)}{8} - \sin^{-1}\frac{\surd 21 - 1 - \surd(6\surd 21 - 18)}{16}.$$

§ 8. The values of sin x, cos x, sin z. cos z, are known to be*

$$\lambda = \sin x = k^3 \frac{\mathrm{cn}^4 \frac{4iK'}{3}}{\mathrm{dn}^4 \frac{4iK'}{3}},$$

$$\lambda' = \cos x = k'^3 . \frac{1}{\mathrm{dn}^4 \frac{4iK'}{3}};$$

$$\mu = \sin z = k^3 \frac{\mathrm{cn}^4 \frac{4K}{3}}{\mathrm{dn}^4 \frac{4K}{3}},$$

$$\mu' = \cos z = k'^3 \frac{1}{\mathrm{dn}^4 \frac{4K}{3}}; \quad \ldots \quad \ldots \quad (8)$$

where the modulus of the elliptic functions is equal to sin y.

From (1 . 2), (1 . 3), (2 . 1) (2 . 2), we have

$$\sin \frac{x+y}{2} = (\lambda k)^{\frac{1}{4}} = + k \frac{\mathrm{cn} \frac{4iK'}{3}}{\mathrm{dn} \frac{4iK'}{3}}$$

$$\cos \frac{x+y}{2} = (\lambda' k')^{\frac{1}{4}} = - k' . \frac{1}{\mathrm{dn} \frac{4iK'}{3}}$$

$$\sin \frac{y+z}{2} = (k \mu)^{\frac{1}{4}} = - k \frac{\mathrm{cn} \frac{4K}{3}}{\mathrm{dn} \frac{4K}{3}}$$

$$\cos \frac{y+z}{2} = (k' \mu')^{\frac{1}{4}} = + k' . \frac{1}{\mathrm{dn} \frac{4K}{3}} \quad \ldots \quad (8.1)$$

It is easily seen that

$$2 \,\mathrm{cn} \frac{4K}{3} \mathrm{dn} \frac{4K}{3} = - \left(1 - k^2 \mathrm{sn}^4 \frac{4K}{3}\right)$$

$$\mathrm{cn}^2 \frac{4K}{3} \left(1 + 2 \,\mathrm{dn} \frac{4K}{3}\right) = \mathrm{sn}^2 \frac{4K}{3} \mathrm{dn}^2 \frac{4K}{3} \quad \ldots \quad (8.2)$$

$$\mathrm{dn}^2 \frac{4K}{3} \left(1 + 2 \,\mathrm{cn} \frac{4K}{3}\right) = k^2 \mathrm{sn}^2 \frac{4K}{3} \mathrm{cn}^2 \frac{4K}{3} \quad \ldots \quad (8.3)$$

* *Vide* Cayley: *Elliptic Functions*, pp. 263 & 268.

Putting x for $\text{dn}\,\frac{4K}{3}$, $\text{cn}\,\frac{4K}{3}$ respectively in (8.2) and (8.3), we obtain the equations

$$x^4 + 2x^3 - 2k'^2 x - k'^2 = 0 \quad \ldots \quad \ldots \quad (8.4)$$

and

$$x^4 + 2x^3 + 2\frac{k'^2}{k^2}x + \frac{k'^2}{k^2} = 0 \quad \ldots \quad (8.5)$$

whose roots are $\text{dn}\,\frac{4K}{3}$, $\text{dn}\,\frac{4iK'}{3}$, $\text{dn}\,\frac{4K+4iK'}{3}$, $\text{dn}\,\frac{-4K+4iK'}{3}$;

and $\text{cn}\,\frac{4K}{3}$, $\text{cn}\,\frac{4iK'}{3}$, $\text{cn}\,\frac{4K+4iK'}{3}$, $\text{cn}\,\frac{-4K+4iK'}{3}$,

respectively

If a, b, c, d are the roots of the equation

$$x^4 - px^3 + qx^2 - rx + s = 0,$$

the reducing cubic whose roots are $ab + cd$, $bc + ad$, $ca + bd$, is

$$x^3 - qx^2 + (pr - 4s)x - (r^2 + p^2 s - 4s^2 q) = 0.$$

The reducing cubics of (8.4) and (8.5) are

$$x^3 + 4k^2k'^2 = 0, \quad \ldots \quad \ldots \quad (8.6)$$

and

$$x^3 - \frac{4k^2k'^2}{k^6} = 0. \quad \ldots \quad \ldots \quad (8.7)$$

Denoting the elliptic functions of arguments $\frac{4K}{3}$, etc. by c_1, c_2, c_3, c_4; and d_1, d_2, d_3, d_4; and writing Ω for $(2kk')^{\frac{2}{3}}$ as before, we obtain the following relations from (8.4), (8.5), (8.6) and (8.7):

$$d_1 + d_2 + d_3 + d_4 = c_1 + c_2 + c_3 + c_4 = -2 \quad \ldots \quad (8.8)$$

$$d_1 d_2 d_3 d_4 = -k^2 c_1 c_2 c_3 c_4 = -k'^2 \quad \ldots \quad (8.81)$$

$$d_1 d_2 + d_3 d_4 = -k^2(c_1 c_2 + c_3 c_4) = -\Omega^2 \quad \ldots \quad (8.82)$$

$$(d_1 + d_2)(d_3 + d_4) = -k^2(c_1 + c_2)(c_3 + c_4) = \Omega^2; \quad (8.83)$$

we have also

$$c_3 + c_4 = 2c_1 c_2 \div 1 - k^2 s_1^2 s_2^2$$

$$d_3 + d_4 = 2d_1 d_2 \div 1 - k^2 s_1^2 s_2^2;$$

whence

$$\frac{c_3 + c_4}{c_1 c_2} = \frac{d_3 + d_4}{d_1 d_2}. \quad \ldots \quad \ldots \quad (8.84)$$

From the relations (8.8) it is easy to obtain

$$\frac{c_1 + c_2}{d_1 d_2} = \frac{c_3 + c_4}{d_3 d_4} = \frac{2}{\Omega^2} \quad \ldots \quad \ldots \quad (8.9)$$

$$\frac{d_1 + d_2}{c_1 c_2} = \frac{d_3 + d_4}{c_3 c_4} = -\frac{2k^2}{\Omega^2}. \quad \ldots \quad (8.91)$$

Jacobi's imaginary transformation, which is equivalent to the substitutions:

$$(k,k'),\ (\mathrm{K},\mathrm{K}'),\ \left(c_1,\ \frac{1}{c_2}\right),\ \left(d_1,\ \frac{d_2}{c_2}\right),\ \left(d_2,\ \frac{d_1}{c_1}\right),$$

denoting geometrically the change of origin from A to B in figure 2, leaves (8 . 9) unaltered, while it changes (8 . 91) into

$$c_1 d_2 + c_2 d_1 = \frac{-2k'^2}{\Omega^2}. \qquad \ldots \qquad \ldots \quad (8.92)$$

The results (8 . 9), (8 . 91), (8 . 92) correspond respectively to those given in (4 . 5), (4 . 3), (4 . 4).

§ 9. To determine $\sin 2x = 2\lambda\lambda' = 2k^3k'^3\dfrac{c_2^{\ 4}}{d_2^{\ 8}}$,

and $\sin 2z = 2\mu\mu' = 2k^3k'^3\dfrac{c_1^{\ 4}}{d_1^{\ 8}}$,

we have to evaluate $\dfrac{c_1}{d_1^{\ 2}}$ and $\dfrac{c_2}{d_2^{\ 2}}$.

Let $$\frac{c_1}{d_1^{\ 2}} = \frac{c_1}{k'^2 + k^2 c_1^{\ 2}} = x.$$

From (8 . 5), c_1 satisfies the relation

$$c_1^{\ 4} + 2c_1^{\ 3} + 2\frac{k'^2}{k^2}c_1 + \frac{k'^2}{k^2} = 0;$$

that is,
$$k^2 c_1^{\ 2} - \frac{c_1}{x} + k'^2 = 0, \qquad \ldots \quad (9)$$

$$k^2 c_1^{\ 4} + 2\frac{c_1^{\ 2}}{x} + k'^2 = 0. \qquad \ldots \quad (9.1)$$

The elimination of c_1 from (9) and (9 . 1) gives the equation

$$x^4 - 4x^3 + \frac{8x}{\Omega^6} + \frac{4}{\Omega^6} = 0, \qquad \ldots \quad (9.2)$$

whose roots are $\dfrac{c_1}{d_1^{\ 2}}$, $\dfrac{c_2}{d_2^{\ 2}}$, $\dfrac{c_3}{d_3^{\ 2}}$, and $\dfrac{c_4}{d_4^{\ 2}}$.

The reducing cubic of (9 . 2) is

$$x^3 - \frac{48x}{\Omega^6} - \left(\frac{64}{\Omega^{12}} + \frac{64}{\Omega^6}\right) = 0. \qquad \ldots \quad (9.3)$$

The left-hand side of (9 . 3) is the product of

$$x - 4\left(\frac{1}{\Omega^2} + \frac{1}{\Omega^4}\right)$$

and
$$x^2 + 4x\left(\frac{1}{\Omega^2} + \frac{1}{\Omega^4}\right) + \frac{16}{\Omega^4}\left(1 - \frac{1}{\Omega^2} + \frac{1}{\Omega^4}\right),$$

whence
$$\frac{c_1 c_2}{d_1^2 d_2^2} + \frac{c_3 c_4}{d_3^2 d_4^2} = 4\left(\frac{1}{\Omega^2} + \frac{1}{\Omega^4}\right). \quad \ldots (9 . 4)$$

Also, from (9 . 2)
$$\frac{c_1 c_2 c_3 c_4}{d_1^2 d_2^2 d_3^2 d_4^2} = \frac{4}{\Omega^6},$$

$$\left(\frac{c_1}{d_1^2} + \frac{c_2}{d_2^2}\right)\left(\frac{c_3}{d_3^2} + \frac{c_4}{d_4^2}\right) = -4\left(\frac{1}{\Omega^2} + \frac{1}{\Omega^4}\right), \quad (9 . 5)$$

$$\frac{c_1}{d_1^2} + \frac{c_2}{d_2^2} + \frac{c_3}{d_3^2} + \frac{c_4}{d_4^2} = 4 .$$

Hence we have,

$$\frac{c_1}{d_1^2} + \frac{c_2}{d_2^2} = 2 - 2\sqrt{1 + \Omega^{-2} + \Omega^{-4}}, \quad \ldots (9 . 6)$$

and
$$\frac{c_1 c_2}{d_1^2 d_2^2} = 2\Omega^{-2} + 2\Omega^{-4} - 2\sqrt{\Omega^{-4} + \Omega^{-6} + \Omega^{-8}}, \ldots (9 . 7)$$

giving
$$\frac{c_1 c_2}{d_1^2 d_2^2}\,\Omega^2 - \left(\frac{c_1}{d_1^2} + \frac{c_2}{d_2^2}\right) = 2\,\Omega^{-2}. \quad \ldots (9 . 8)$$

The results (9 . 6), (9 . 7) and (9 . 8) are the same as (6 . 2), (6 . 3) and (6 . 1) with $p = -\frac{c_2}{d_2^2}$, and $q = -\frac{c_1}{d_1^2}$ respectively.

Notes and Questions.

Common Normals to Two Algebraic Curves.

1. *Two curves of degrees n, n' and classes m, m' pass g, g' times through a circular point ; and the line at infinity touches them f, f' times respectively. To find the number of common normals other than the line at infinity.*

A common normal to two curves is evidently a common tangent to their evolutes ; so that, if m_1, m_2 denote the classes of the evolutes of the curves, the number of common normals to the curves will be $m_1 m_2$.

Now $$m_1 = m + n - f - g, \quad m_2 = m' + n' - f' - g'. \quad \ldots \quad (1)$$

The line at infinity cuts the curves in n, n' points respectively and the normal at any such intersection to the corresponding curve coincides with the line at infinity itself, unless the point is a circular point. Hence it follows that the line at infinity corresponds to nn' common normals, provided that none of its intersections with the curves is a circular point. If however g, g' intersections coincide with the circular points, then the line at infinity corresponds to $(n - g)(n' - g')$ common normals. Thus the number of common normals (real or imaginary) other than the line at infinity, is N, where

$$N = [(m+n-f-g)(m'+n'-f'-g')-(n-g)(n'-g')]. \quad \ldots \quad (2)$$

Particular Cases : Let the curves be

(i) two central conics, $N = 12$;

(ii) one central conic and one parabola, $N = 8$;

(iii) two parabolas, $N = 5$;

(iv) one central conic and one circle, $N = 8$;

(v) one parabola and one circle, $N = 6$;

(vi) two non-concentric circles, $N = 4$.

(A diameter of a circle being a double normal, the line joining the centres corresponds to four normals).

[*N.B.*—The number of common normals to two *parallel* curves is infinite, because they have the same evolute. Hence an infinite number of normals can be drawn to two concentric circles.]

2. We verify the cases of (i) a parabola and its evolute, (ii) an ellipse and its evolute.

(i) If the equation of a parabola be $y^2 = 4\mu(x + 2\mu)$, then the tangential equation of the evolute is

$$\mu\xi^3 + \eta^2 = 0, \qquad \ldots \qquad \ldots \qquad (3)$$

and that of the evolute of the evolute is

$$\xi^3 + 2\mu\eta^4 + 3\mu\xi^2\eta^2 = 0. \qquad \ldots \qquad \ldots \qquad (4)$$

Solving (3) and (4) we find that a parabola and its evolute have six common normals.

Now for the parabola

$$n = 2,\ m = 2,\ f = 1,\ g = 0;$$

and for its evolute

$$n' = 3,\ m' = 3,\ f' = 2,\ g' = 0.$$

Hence by the formula also we have N = 6.

Eliminating η we get

$$2\mu^3\xi^3 - 3\mu^2\xi^2 + 1 = 0,$$

which gives

$$\mu\xi = 1,\ 1,\ -\tfrac{1}{2}, \quad \therefore\ \mu\eta = \pm i,\ \pm i,\ \pm\frac{1}{2\sqrt{2}}.$$

Hence each circular line through the focus corresponds to a double normal, as is otherwise obvious; and the other common normals are

$x \pm \frac{y}{\sqrt{2}} = 2\mu$ at the points $(0, \pm 2\sqrt{2}\mu)$ of the parabola $y^2 = 4\mu(x + 2\mu)$,

and the points $\left(\frac{3}{2}\mu, \pm\frac{1}{\sqrt{2}}\mu\right)$ of its evolute

$$27\mu y^2 = 4x^3;$$

and they are tangents to the latter curve at the points

$$(6\mu, \mp 4\sqrt{2}\mu).$$

Now, a common normal to the parabola and its evolute meets the latter at a point the tangent at which is normal to the parabola. In other words, that point is a point of intersection of two perpendicular normals to the parabola and must therefore lie on the locus of the intersection of two perpendicular normals, *viz.*

$$y^2 = \mu(x - \mu).$$

Further, it is obvious that the evolute should touch the above locus at the point in question, since the point is a point of intersection of consecutive normals of the original parabola. Hence the point may be geometrically defined as the point of contact of the evolute with the above locus.

The distance between the feet of a common normal is

$$r = \frac{3\sqrt{3}}{2}\mu.$$

The centres of curvature corresponding to the feet are

$$(6\mu, \ \mp 4\sqrt{2}\mu) \text{ and } (-3\mu, \ \pm 5\sqrt{2}\mu);$$

and the radii of curvature at the feet are given by

$$\rho_1 = 6\sqrt{3}\mu, \ \rho_2 = \frac{9\sqrt{3}}{2}\mu. \ \therefore \rho_1 - \rho_2 = \frac{3\sqrt{3}}{2}\mu = r.$$

Hence we get:

The difference between the radii of curvature at the feet of a common normal to a parabola and its evolute is equal to the distance between the feet.

(ii) If the equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

that of its evolute is $(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = c^{\frac{4}{3}}$, where $c^2 = a^2 - b^2$.

The tangential equations of their evolutes are

$$c^4\xi^2\eta^2 = a^2\eta^2 + b^2\xi^2, \ c^4(a^2\xi^4 - b^2\eta^4)^2 = (a^2\xi^2 + b^2\eta^2)^3.$$

Eliminating η^2 and putting $c^2\xi^2 = z$, we get

$$(z-1)^2 = 0,$$

and $a^2c^4z^3 - a^2c^2(3a^2+2b^2)z^2 + (3a^6+4a^4b^2-2b^6)z - (a^2+b^2)^3 = 0.$ (1)

Now $z=1$ gives the circular lines through the foci as common normals, each circular line being a double normal, and the 2nd equation above gives twelve common normals to the ellipse and its evolute since to each value of ξ, there correspond two values of η.

Hence the number of finite common normals is twenty and the same result is given by the formula.

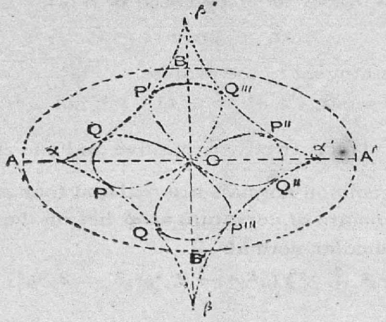
In the process of elimination, we rejected ξ^6; and $\xi^6 = 0$, gives twelve normals coincident with the line at infinity.

The locus of the intersection of two perpendicular normals to the ellipse is the sextic

$$(a^2 + b^2)(x^2 + y^2)(a^2y^2 + b^2x^2)^2 = (a^2 - b^2)^2(a^2y^2 - b^2x^2)^2$$

which intersects the evolute of the ellipse in thirty-six points, four of which are the points of contact, with the evolute, of the circular lines through the foci.

Here also, as in the case of the parabola, a real common normal to the ellipse and its evolute meets the latter at a point of contact with the above sextic.



Analytically, this may be seen as under:

Two normals

$$ax \sec\theta - by \operatorname{cosec}\theta = c^2,\ ax \sec\phi - by \operatorname{cosec}\phi = c^2$$

are perpendicular to each other if

$$\tan\theta \tan\phi = -\frac{b^2}{a^2} \qquad \ldots \qquad \ldots \quad (2)$$

The centre of curvature at ϕ, *viz.*

$$\left(\frac{c^2}{a}\cos^3\phi,\ -\frac{c^2}{b}\sin^3\phi\right)$$

lies on the normal at θ, if

$$\cos^3\phi \sec\theta + \sin^3\phi \operatorname{cosec}\theta = 1. \qquad \ldots \quad (3)$$

Eliminating θ from (2), (3), we get

$$a^8z^5 - a^4b^2(2a^2+3b^2)z^3 - a^2b^4(2b^2+3a^2)z^2 + b^8 = 0$$

where $z = \tan^2\phi$.

Since $\tan\phi = \tan(\pi + \phi)$, it gives twenty common normals; the factor $(a^2z + b^2) = 0$ gives circular lines through the foci as double common normals. The equation

$$a^4z^3 - 2a^2b^2z^2 - 2a^2b^2z + b^4 = 0 \quad \ldots \quad \ldots (4)$$

gives the other twelve common normals.

It is easy to see that (1) is reducible to (4), if we put in (1)

$$c^2z = a^2(1 + t^2), \text{ where } \mu^2t^2a^4 = b^4.$$

Now $$ax \sec\theta - by \operatorname{cosec}\theta = c^2,$$

and $$x\xi + y\eta + 1 = 0,$$

represent the same tangent to the evolute.

$$\therefore \quad c^2\xi = a\sec\theta, \text{ i.e. } c^2z = a^2(1+t^2), \text{ but } \mu^2t^2a^4 = b^4.$$

The equation (4) has two roots positive and one negative.

Hence eight common normals are real and four imaginary. Further we notice that the centre of curvature at ϕ lies on the locus of intersection of two perpendicular normals, if

$$(a^2z + b^2)(a^2z^2 + b^2)(a^4z^3 - 2a^2b^2z^2 - 2a^2b^2z + b^4)^2 = 0,$$

where $z = \tan^2\phi$.

The factor $a^2z + b^2 = 0$ gives four circular lines; and

$$(a^4z^3 - 2a^2b^2z^2 - 2a^2b^2z + b^4)^2 = 0$$

shows that the locus touches the evolute at the feet of the common normals.

Also $a^2z^2 + b^2 = 0$ gives $\tan\phi = \pm\dfrac{1 \pm i}{\sqrt{2}}\sqrt{\dfrac{b}{a}}$,

corresponding to eight intersections. In this case, the other two normals from the centre of curvature

$$\left[\frac{\sqrt{a}\,c^2}{(a + ib)^{\frac{3}{2}}},\ \frac{(i - 1)\sqrt{b}\,c^2}{\sqrt{2}(a + ib)^{\frac{3}{2}}}\right]$$

i.e. the normals other than the tangent to the evolute at the point are at right angles to each other. So the locus does not touch the evolute at such a centre.

Thus we see that four points of intersections of the sextic and the evolute are the points of contact of the circular lines through the foci with the evolute, twelve are the feet of common normals whereat they touch, and eight are the centres of curvature corresponding to

$$a^2 z^2 + b^2 = 0.$$

Hence all the thirty-six intersections are accounted for.

There are only *eight real* common normals to an ellipse and its evolute; and their feet on the evolute are the points of contact with the sextic already mentioned. Further there are eight imaginary points on the evolute from each of which normals to the ellipse (other than the tangent thereat to the evolute) are perpendicular to each other.

Again each root of the equation

$$a^4 z^3 - 2a^2 b^2 z^2 - 2a^2 b^2 z + b^4 = 0$$

gives four common normals. It follows that twelve finite common normals are grouped into three sets of four.

Let PR be any common normal of one set and PR $= r_1$; and let ρ_1', ρ_1 be the radii of curvature at P, R.

Also, let ρ_2, ρ_2', r_2; ρ_3, ρ_3', r_3 be similar quantities for the other two sets. Then it follows that

$$\rho_1\rho_2\rho_3 = \frac{3\sqrt{3}\, a^6 b^6}{(a^2+b^2)^{\frac{9}{2}}}; \quad \rho_1'\rho_2'\rho_3' = \frac{81\sqrt{3}\, a^2 b^2 (a^2-b^2)^2}{(a^2+b^2)^{\frac{5}{2}}};$$

$$r_1 r_2 r_3 = -\frac{3\sqrt{3}\, a^2 b^2}{(a^2+b^2)^{\frac{1}{2}}};$$

$$(\rho_1 + r_1)(\rho_2 + r_2)(\rho_3 + r_3) = \tfrac{1}{9}\rho'_1\rho'_2\rho'_3,$$

$$(\rho_1 - r_1)(\rho_2 - r_2)(\rho_3 - r_3) =$$

$$\frac{3\sqrt{3}a^2b^2[5(a^2+b^2)^4 - 4a^2b^2(a^2+b^2)^2 + 4a^4b^4]}{(a^2+b^2)^{\frac{9}{2}}}.$$

HEMRAJ.

Note on a Formula in Solid Geometry.

The axes being oblique to find the angle between two directions whose ratios (x_1, y_1, z_1), (x_2, y_2, z_2) *are given by the equations*

$$T \equiv ux + vy + wz = 0, \qquad \ldots (1)$$

$$S \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0. \qquad \ldots (2)$$

With reference to the solution of this problem communicated to the *Journal* (June 1923) by Prof. E. H. Neville, the following additional solutions should be interesting :—

Notation: α, β, γ are the angles of the given frame, and θ the required angle.

Δ, ϕ, Δ' are the co-efficients of k^2, k, k° in the equation

$$\begin{vmatrix} ka+1 & kh+\cos\gamma & kg+\cos\beta & u \\ kh+\cos\gamma & kb+1 & kf+\cos\alpha & v \\ kg+\cos\beta & kf+\cos\alpha & kc+1 & w \\ u & v & w & o \end{vmatrix} = 0.$$

$$S_{pq} \equiv \Sigma a x_p x_q + \Sigma f(y_p z_q + y_q z_p).$$

$$\sigma \equiv \Sigma x^2 + 2\Sigma yz\cos\alpha.$$

$$\sigma_{pq} \equiv \Sigma x_p x_q + \Sigma (y_p z_q + y_q z_p)\cos\alpha.$$

First Solution:

The values of k, for which the equations

$$T = 0 \text{ and } kS + \sigma = 0 \qquad \ldots (3)$$

give coincident directions, are the roots of the equation

$$k^2\Delta + k\phi + \Delta' = 0. \qquad \ldots (4)$$

Now, if the directions of reference be changed, and the directions given by (1) and (2) be taken for

$$y = 0,\ x = 0 \text{ and } z = 0,\ x = 0,$$

and if θ, $\frac{\pi}{2}$, $\frac{\pi}{2}$ be the angles of the new frame, the equations (3) are transformed into $x = 0$ and

$$k(a_1 x^2 + 2\Sigma f_1 yz) + (\Sigma x^2 + 2yz\cos\theta) = 0.$$

The equation (4) therefore becomes

$$k^2 f_1^2 + 2 k f_1 \cos \theta - \sin^2 \theta = 0.$$

$$\therefore \quad \frac{f_1^2}{\Delta} = \frac{2 f_1 \cos \theta}{\phi} = -\frac{\sin^2 \theta}{\Delta'}$$

$$\therefore \quad \tan^2 \theta = -\frac{4 \Delta \Delta'}{\phi^2}.$$

Second Solution:

From the usual formula $\cos^2 \theta = \dfrac{\sigma_{12}^2}{\sigma_{11}\sigma_{22}}$,

$$\tan^2 \theta = \frac{\sigma_{11}\,\sigma_{22} - \sigma_{12}^2}{\sigma_{12}^2}. \qquad \ldots(5)$$

Now the ratios $(\lambda x_1 + x_2, \lambda y_1 + y_2, \lambda z_1 + z_2)$ satisfy the equation $T = 0$ for all values of λ.

Hence, remembering that $S_{11} = S_{22} = 0$, the values of k, for which the equations

$$T = 0 \text{ and } kS + \sigma = 0$$

give coincident directions, are the roots of the equation

$$k^2 S_{12}^2 + 2k\, S_{12}\, \sigma_{12} - (\sigma_{11}\,\sigma_{22} - \sigma_{12}^2) = 0.$$

They are also given by the equation

$$k^2 \Delta + k\phi + \Delta' = 0.$$

$$\therefore \quad \frac{S_{12}^2}{\Delta} = \frac{2\, S_{12}\, \sigma_{12}}{\phi} = -\frac{\sigma_{11}\,\sigma_{22} - \sigma_{12}^2}{\Delta'}.$$

$$\therefore \quad \tan^2 \theta = -\frac{4 \Delta \Delta'}{\phi^2}.$$

NOTE:—Either method is applicable to the problem of calculating the angle between two vectors whose co-efficients satisfy simultaneously $(n - 2)$ linear equations and a homogeneous quadratic. The value of $\tan^2\theta$ is $-\dfrac{4\,DD'}{\psi^2}$, where D, D', ψ are the co-efficients of a certain quadratic equation.

F. H. V. GULASEKHARAM.

Solutions.

Question 469.

(S. RAMAÑUJAM) :—The number $\{n! + 1\}$ is a perfect square for the values 4, 5, 7 of n. Find other values.

Remarks by M. Bhimasena Rao.

This was proposed in the year 1913. The following note relating to the question is taken from Mr. L. E. Dickson's *History ef the Theory of Numbers*, Vol. II. :—

In 1876 H. Brocard asked in *Nouv. Corresp. Math.*, 2, 287 ; for values of x making $1 + x!$ a square. Nine years after in 1885, he suggested in *Nouv. Ann. Math.*, 4, 391, that the only solutions are 4, 5, 7. A. Gerardin remarked in the latter journal of 1906 that if $1 + x! = y^2$ has solutions other than $x = (4, 5, 7)$; $y = (5, 11, 71)$, then y has at least 20 digits.

Question 1205.

(A. A. KRISHNASWAMI AIYANGAR):—If n be an even number, prove that all the non-repeated factors of $(2^n - 1)$ are prime to n and that the repeated factors occur once oftener than in n.

Solution by K. Satyanarayana.

Let $n = 2rp^\alpha$, p being a prime, and r and p being prime to one another.

Then $2^{2rp^\alpha} - 1 = (2^p)^{2rp^{\alpha-1}} - 1 = [2\,(2^{p-1})\,]^{2rp^{\alpha-1}} - 1$

$= M(p) + 2^{2rp^{\alpha-1}} - 1$, by Fermat's theorem.

Hence it follows that, if p is a factor of $2^{2rp^\alpha} - 1$, p should be a factor of

$$2^{2rp^{\alpha-1}} - 1,\ 2^{2rp^{\alpha-2}} - 1,\ \ldots\ldots,\ 2^{2rp} - 1,\ 2^{2r} - 1.$$

But if $2^{2r} - 1 = kp$, then

$$2^{2rp} - 1 = (kp + 1)^p - 1 = M(p^2),$$

$$22rp^2 - 1 = M(p^3)$$

and $$2^{2rp\alpha} - 1 = M(p^{\alpha+1}).$$

This proves that if a prime p be a common factor in n and $2^n - 1$, then p occurs once oftener in $(2^n - 1)$ than in n. Consequently non-repeated prime factors of $2^n - 1$ cannot occur in n.

[*Remarks* : The following very important point has to be noted in connection with the solution :—

It is assumed that k is prime to p, for if k contain p once as a factor, $(2^{2r} - 1)$ contains p^2 as a factor and therefore $(2^n - 1)$ contains $p^{(\alpha+2)}$.

It has to be established therefore that if r and p be prime to one another $2^{2r} - 1$, if it contains p as a factor, can have it only once.

This result follows, if it can be shown that $2^{p-1} - 1$ cannot contain p^2 as a factor, p being a prime number.

For suppose $2^{p-1} - 1 = ap$ where a is prime to p. Then

$$2^{2r(p-1)} - 1 = (ap + 1)^{2r} - 1 = bp,$$

where b is prime to p. If $2^{2r} - 1$ contains p as a factor, let

$$2^{2r} - 1 = cp.$$

Then $$2^{2r(p-1)} - 1 = (cp + 1)^{p-1} - 1 = dp,$$

where d will be prime to p, provided c is prime to p. But $b = d$ and we know that if a is prime to p, b is prime to p; and that if b or d is to be prime to p, c should be prime to p.

So the problem is correct or not, according as, p being an odd prime, $2^{p-1} - 1$ is not or is divisible by p^2.]

Question 1223.

(A. A. KRISHNASWAMI AIYANGAR) :—Prove that the orthoptic locus of the first negative pedal of an ellipse with respect to any point on its circumference, is a similar ellipse.

Solution by S. Audinarayanan.

Let OPP′ be the ellipse and O a fixed point on it. Let OP and OP′ be any two chords at right angles. Complete the rectangle OPTP′ and let T′ be the midpoint of its diagonals.

Now the first negative pedal of the ellipse is the envelope of the straight line perpendicular to the radius vector through a point on the ellipse; that is, the envelope of straight lines like PT and P′T. The orthoptic locus of a curve is the locus of points from which tangents to the curve are at right angles.

The orthoptic locus is thus the locus of points like T, since PT and P′T are one set of perpendicular tangents.

We shall prove that the locus of T is a similar conic.

Since PP′ always subtends a right angle at the fixed point O, it always passes through a fixed point on the normal at O.

The locus of mid points of chords (of an ellipse) passing through a fixed point is a similar ellipse. So the locus of T′ is an ellipse similar to the given ellipse.

Again OT′ : OT = 1 : 2. Hence the locus of T is an ellipse similar to the given ellipse.

Question 1227.

(PROF. K. J. SANJANA):—A heavy homogenous string is tightly wrapped round a plane vertical curve which is such that the pressure of the curve per unit length varies as the weight of unit length of the string. Prove that the intrinsic equation of the curve is

$$s = cl - \frac{l \sin \phi}{(k^2-1)(k+\cos \phi)} + \frac{kl}{(k^2-1)^{3/2}} \cos^{-1} \frac{1+k\cos \phi}{k+\cos \phi},$$

where k is the constant of variation and l stands for $kt_0 - mg$, m being the mass of the string per unit length and t_0 the tension at the point where the tangent to the curve is vertical. [When k is < 1, the last term contains the inverse hyperbolic cosine.]

When the pressure is equal to the weight, prove that

$$s = \frac{1}{6} l \left(c + 3 \tan \frac{\phi}{2} + \tan^3 \frac{\phi}{2} \right)$$

and show that the curve is then the first negative pedal of a parabola with respect to the focus.

Solution by K. Satyanarayana, S. Audinarayanan and M. V. Ramakrishnan.

Resolving along the tangent and normal the forces on an element, δs, we have by the problem,

$$\frac{dT}{ds} = mg \sin \psi,$$

and

$$\frac{T}{\rho} = mg \cos \psi + R = mg(k + \cos \psi).$$

$$\therefore \quad T = mg\, \rho\, (k + \cos \psi).$$

$$\therefore \quad \frac{dT}{ds} = mg \left[\frac{d\rho}{d\psi} \cdot \frac{1}{\rho}(k + \cos \psi) - \sin \psi \right] = mg \sin \psi.$$

$$\therefore \quad \frac{d\rho}{\rho} = \frac{2 \sin \psi\, d\psi}{k + \cos \psi}, \text{ and } \rho = \frac{A}{(k + \cos \psi)^2} = \frac{T}{mg(k+\cos \psi)}.$$

But when $\psi = \frac{\pi}{2}$, $T = t_0$. $\therefore A = \frac{kt_0}{mg} = l$.

$$\therefore \quad \frac{ds}{d\psi} = \frac{l}{(k + \cos\psi)^2}. \quad \ldots \quad \ldots \quad (1)$$

(i) When $k > 1$

$$s = C - \frac{l \sin\psi}{(k^2 - 1)(k + \cos\psi)} + \frac{kl}{(k^2-1)^{3/2}} \cos^{-1}\left(\frac{1 + k\cos\psi}{k + \cos\psi}\right);$$

(ii) when $k < 1$

$$s = C + \frac{l \sin\psi}{(1 - k^2)(k + \cos\psi)} - \frac{kl}{(1-k^2)^{3/2}} \cosh^{-1}\left(\frac{1 + k\cos\psi}{k + \cos\psi}\right);$$

[*N.B.*—For integration of (1), refer to Loney's *Dynamics of a Particle*, pages 85, 86.]

(iii) when $k = 1$, $\frac{ds}{d\psi} = \frac{l}{(1 + \cos\psi)^2}$. $\quad \ldots \quad \ldots \quad (2)$

$$\therefore \quad s = C + \tfrac{1}{2} l \tan \tfrac{1}{2}\psi + \tfrac{1}{6} l \tan^3 \tfrac{1}{2}\psi;$$

whose (p, r) equation is got as follows:

$$\rho = p + \frac{d^2 p}{d\psi^2} = \frac{l}{(1 + \cos\psi)^2} \text{ by (2)}$$

$$\therefore \quad p = L\cos\psi + M\sin\psi + \frac{1}{D^2 + 1}\frac{l}{(1 + \cos\psi)^2}.$$

$$= L\cos\psi + M\sin\psi + \frac{l}{1 + \cos\psi}.$$

$$\therefore \quad \frac{dp}{d\psi} = -L\sin\psi + M\cos\psi + \frac{l\sin\psi}{(1 + \cos\psi)^2}.$$

Choose the origin such that when

$$\psi = \frac{\pi}{2}, p = l, \text{ and } \frac{dp}{d\psi} = r\frac{dr}{ds} = l;$$

then $\quad L = 0, \ M = 0.$

$$\therefore \quad p = r\sin\phi = \frac{l}{1 + \cos\psi}.$$

$$\frac{dp}{d\psi} = r\cos\phi = \frac{l\sin\psi}{(1 + \cos\psi)^2}.$$

$$\therefore \quad r^2 = \frac{2l^2}{(1 + \cos\psi)^3} = \frac{2}{l}p^3.$$

Hence the pedal equation is $p^3 = \frac{1}{2} l r^2$, which is the first negative pedal of the parabola $p^2 = \frac{1}{2} l r$ with respect to the focus.

Question 1232.

(G. V. T.):—The perpendiculars from ABC on the sides BC, CA, AB of a Δ ABC are divided at L, M, N in the ratio $k:1$. O and H are the circum-centre and the ortho-centre of the triangle. OL, OM, ON meet BC, CA, AB in X, Y, Z respectively. Show that AX, BY, CZ meet at a point on OH. If L′, M′, N′ be the isogonal conjugates of L, M, N with respect to the Δ ABC, show that XL′, YM′, ZN′ intersect at H.

Solution (1) by K. Srinivasa Raghavan, and (2) by P. Krishnamachari.

(1) The normal co-ordinates—

of O are (cos A, cos B, cos C.)

H $\left(\frac{1}{\cos A}, \frac{1}{\cos B}, \frac{1}{\cos C}\right)$

L $(k, \cos C, \cos B)$

M $(\cos C, k, \cos A)$

N $(\cos B, \cos A, k)$.

Let $lx + my + nz = 0$ be the equation of LOX. Then

$$l \cos A + m \cos B + n \cos C = 0$$

and

$$lk + m \cos C + n \cos B = 0.$$

$$\therefore \quad \frac{l}{\cos^2 B - \cos^2 C} = \frac{m}{k \cos C - \cos A \cos B} = \frac{n}{\cos C \cos A - k \cos B}.$$

The equation to AX is $my + nz = 0$.

or

$$\frac{y}{k \cos B - \cos A \cos C} = \frac{z}{k \cos C - \cos A \cos B}.$$

Similarly for BY and CZ, we have

$$\frac{x}{k \cos A - \cos B \cos C} = \frac{z}{k \cos C - \cos A \cos B}$$

and

$$\frac{x}{k \cos A - \cos B \cos C} = \frac{y}{k \cos B - \cos A \cos C}.$$

Thus we see that AX, BY, CZ, are concurrent at

$$\frac{x}{k \cos A - \cos B \cos C} = \frac{y}{k \cos B - \cos C \cos A} = \frac{z}{k \cos C - \cos A \cos B}.$$

Let this point be P. Now the determinant

$$\begin{vmatrix} x & y & z \\ \sec A & \sec B & \sec C \\ \cos A & \cos B & \cos C \end{vmatrix}$$

is zero, which shows that P, H, O are collinear.

The co-ordinates of L′, M′, N′ are

$$\left(\frac{1}{k}, \frac{1}{\cos C}, \frac{1}{\cos B}\right), \left(\frac{1}{\cos C}, \frac{1}{k}, \frac{1}{\cos A}\right), \left(\frac{1}{\cos B}, \frac{1}{\cos A}, \frac{1}{k}\right).$$

as they are the isogonal conjugates of L, M, N. Since the determinant

$$\begin{vmatrix} \frac{1}{k} & \frac{1}{\cos C} & \frac{1}{\cos B} \\ 0, & k\cos B - \cos A\cos C, & k\cos C - \cos A\cos B \\ \frac{1}{\cos A} & \frac{1}{\cos B} & \frac{1}{\cos C} \end{vmatrix}$$

is zero, L′, X, H are collinear. Similarly M′Y, N′Z also pass through H.

Hence XL′, YM′, ZN′ concur at H.

(2) Let AX cut HO in K and let CH cut AB in F.

In the △ALX where KOH is a transversal

$$\frac{XO}{OL} \cdot \frac{LH}{HA} \cdot \frac{AK}{KX} = -1. \qquad \dots \qquad \dots \qquad (1)$$

Remembering that the ⊥r from O on BC = ½ AH, we see that

$$\frac{XO}{OL} = \frac{AH}{2LD - AH}.$$

∴ Substituting in (1) and reducing, regard being paid to the proper sign,

$$\frac{AK}{KX} = \frac{2LD - AH}{LH}.$$

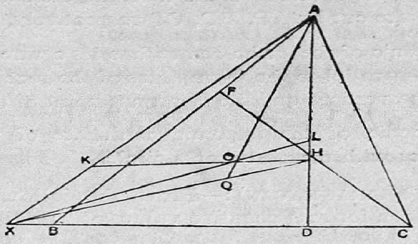
$$\therefore \qquad \frac{AX}{XK} = -\frac{2LD - AL}{LH}. \qquad \dots \qquad (2)$$

In the △ AKH where XOL is a transversal

$$\frac{AX}{XK} \cdot \frac{KO}{OH} \cdot \frac{HL}{LA} = -1. \qquad \dots \qquad \dots \qquad (3)$$

Substituting in (3) from (2) and reducing, we have

$$\frac{KO}{OH} = \frac{AL}{2LD - AL} = \frac{k}{2-k}.$$



∴ By symmetry, we see that BY and CZ also pass through K.

If XH intersects AO in Q, in the △ AOL

$$\frac{AQ}{QO} \cdot \frac{OX}{XL} \cdot \frac{LH}{HA} = -1.$$

Also $$\frac{XO}{XL} = \frac{\frac{1}{2}AH}{LD} = \frac{AH}{2LD}.$$

Substituting and reducing $\frac{AQ}{QO} \cdot \frac{LH}{2LD} = -1.$ ... (4)

In the △ AQH since XOL is a transversal,

$$\frac{QO}{OA} \cdot \frac{AL}{LH} \cdot \frac{HX}{XQ} = -1. \quad \ldots \quad \ldots \quad (5)$$

∴ Multiplying (4) and (5)

$$\frac{AL \cdot AQ}{2AO \cdot LD} \cdot \frac{XH}{XQ} = 1.$$

It is readily seen that the product of ⊥rs from Q and L on AB is equal to the product of ⊥rs from Q and L on BC.

∴ BL and BQ make equal angles with the bisector of the angle B.

Obviously AO or AQ and AH make equal angles with the bisector of the angle A.

∴ L and Q are isogonal conjugates with respect to the △ABC.

∴ L′ and Q coincide, and hence the second part.

Questions for Solution.

1316. (R. GOPALASWAMI) :—Defining *the inconic of P with respect to a triangle ABC*, as the conic touching the sides of ABC at the points where they are cut by AP, BP, CP respectively, prove that

(i) the inconic of the ortho-centre has its centre at the Lemoine point and its axes parallel to the asymptotes of the rectangular hyperbola passing through the vertices and the Gergonne point of the pedal triangle.

(ii) the inconic of the isotomic conjugate of the circumcentre has double contact with the nine-points-circle.

1317. (T. VIJIARAGHAVAN) :—Prove the following properties of the Gamma function for real negative values of x :—

(i) $|\Gamma(x)|$ has only one minimum in the interval $(-n, -n+1)$ where n is a positive integer.

(ii) If t_n is the number between $(-n, -n+1)$ for which $|\Gamma(x)|$ is a minimum, then $(t_n + n)$ tends steadily to zero and $(t_n + n)\log n$ approaches 1.

(iii) $\Gamma(t_n) \sim e \log n \,[n!]^{-1}$

(iv) If $\epsilon_n \geqslant 1/\sqrt{(n-1)!}$, then, on the graph of $\Gamma(x)$ in the range $(-n, -n+1)$, the distance of any point to one of the lines $y=0$, $x=-n$, $x=-n+1$ is less then ϵ_n.

1318. (V. V. SATHYANARAYANA MURTHY) :—If the diagonals of a cyclic quadrilateral are conjugate lines and a homothetic quadrilateral be described with their intersection as homothetic centre, prove that the consecutive pairs of sides of the one quadrilateral intersect the corresponding pairs of the other in eight points which lie on a circle co-axal with the circum-circles of the quadrilaterals.

1319. (B. B. BAGI) :—ABC is a spherical triangle. Any point P of the sphere is joined to ABC and the cosines of these arcs are denoted by (a_1, b_1, c_1); similarly $(a_2b_2c_2)$ $(a_3b_3c_3)$ $(a_4b_4c_4)$ denote the cosines of the arcs joining three other points Q, R, S to the vertices of ABC. Prove that the four points P, Q, R, S lie on a small circle, if

$$\begin{vmatrix} a_1 & b_1 & c_1 & 1 \\ a_2 & b_2 & c_2 & 1 \\ a_3 & b_3 & c_3 & 1 \\ a_4 & b_4 & c_4 & 1 \end{vmatrix} = 0.$$

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LIST OF JOURNALS RECEIVED

(from 20th of October 1923 to 2nd of January 1924)

1 Nature, Vol. 112. Nos. 2813, 2814, 2815, 2816, 2817, 2818.

2 American Journal of Mathematics, Vol. 43, No. 4.

3 Philosophical Magazine, Vol. 46, Nos. 274, 275, 276.

4 Messenger of Mathematics, Vol. 53, Nos. 1, 2.

5 Popular Astronomy, Vol. 30, No. 6; Vol. 31, No. 8. (Two copies each).

6 Bulletin of the American Mathematical Society. Vol. 29, Nos. 8, 9. (Two copies each).

7 Annals of Mathematics, Vol. 24, No. 2.

8 Transactions of the American Mathematical Society, Vol. 25, Nos. 2 & 3.

9 Astrophysical Journal, Vol. 57, Nos. 1, 2, 3.

10 Proceedings of the Royal Society (London), Series A, Vol. 104, Nos. 725, 726, 727.

11 Transactions of the Cambridge Philosophical Society, Vol. 23, No. 1.

12 Philosophical Transactions of the Royal Society of London, Series A, Vol. 223 (Title, Contents and Index issue).

13 Proceedings of the London Mathematical Society, Series 2, Vol. 22, Parts 2, 3.

14 Mathematical Gazette, Vol. 11, No. 166. (Two copies).

15 The American Mathematical Monthly, Vol. 23, No. 6 (Sept.-Oct.), No. 7 (Nov.)

16 The Journal of the Science Association, Vizianagaram, Vol. 1, Nos. 1, 2 & 3.

17 Monthly Notices of the Royal Astronomical Society, Vol. 83, No. 9, Oct. 1923.

18 Acta Mathematica, Vol. 39, Nos. 1 & 2.

19 Bulletin of the Calcutta Mathematical Society, Vol. 14, No. 2.

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